A note on a secondary Pohlke's projection

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Abstract

Given three segments OP_1, OP_2, OP_3 in a plane ω, which are not contained in a line, we find a simple necessary and sufficient condition for the existence of two distinct ellipses centered at O and circumscribing the three ellipses having as conjugate semi-diameters the pairs (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1). We prove this result by showing that it is equivalent to the existence of a *secondary Pohlke's projection* closely related to the (always existing) projection given by Pohlke's fundamental theorem of oblique axonometry.

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1 Introduction

The presents paper is inspired by Toulias and Lefkaditis's article [8], where the authors investigated from the analytic plane geometry's point of view the problem of determining all the concentric ellipses circumscribing a given set of three *mutually conjugated* ellipses.

This question arises from the fact that the existence of a circumscribing ellipse is an intermediate step in some of the proofs of Pohlke's fundamental theorem of axonometry [6]. See [1], [2], [3] and the references therein. Conversely, the existence of such an ellipse is also an immediate consequence of this same theorem.

More precisely, let ω be a plane in the Euclidean space $\mathbb{E}^3$ and let $OP_1, OP_2, OP_3 \subset \omega$ be three segments which are not contained in a line. By Pohlke's theorem we know that *there are a parallel projection $\Pi : \mathbb{E}^3 \to \omega$ and three equal segments OQ_1, OQ_2, OQ_3 such that*

$$\Pi(OQ_i) = OP_i \quad (1 \le i \le 3) \tag{1.1}$$

and

$$OQ_1 \perp OQ_2,\ OQ_2 \perp OQ_3,\ OQ_3 \perp OQ_1. \tag{1.2}$$

Moreover, Π is unique up to symmetry with respect to ω.[1] *The set of three segments OQ_1, OQ_2, OQ_3 is determined up to symmetry with respect to ω and up to symmetry with respect to a plane through O and perpendicular to the direction of projection.*[2] See [1], [4].

[1] The symmetrical projection $\bar{\Pi} : \mathbb{E}^3 \to \omega$, defined by $\bar{\Pi}(P) = \Pi(\bar{P})$ where $\bar{P}$ is symmetric to P with respect to ω, satisfies both (1.1) and (1.2) with $\bar{Q}_1, \bar{Q}_2, \bar{Q}_3$ instead of Q_1, Q_2, Q_3.

[2] The conditions (1.1), (1.2) continue to apply if we replace Q_1, Q_2, Q_3 with their symmetrical Q_1', Q_2', Q_3' with respect to the plane through O and perpendicular to the direction of projection.

Now, if OP_1, OP_2, OP_3 are non-parallel, we may consider (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1) as pairs of conjugate semi-diameters of three concentric ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$ respectively. So a consequence of the previous statement is that:

There exits an ellipse $\mathcal{E}$ centered at O and circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

Indeed, let S be the sphere centered at O and containing Q_1, Q_2, Q_3. By (1.1), (1.2), the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ are the projections, via Π, of the great circles of S through the pairs (Q_1, Q_2), (Q_2, Q_3) and (Q_3, Q_1) respectively. So it is enough to consider the ellipse $\mathcal{E}$ formed from projecting onto ω the great circle of S in the plane π through O and perpendicular to the direction of projection. Namely, $\mathcal{E} = \Pi(S \cap \pi)$.

Definition 1.1 *Let $\Pi : \mathbb{E}^3 \to \omega$ be a parallel projection. We call Π a Pohlke's projection for OP_1, OP_2, OP_3 if we can find equal segments OQ_1, OQ_2, OQ_3 such that (1.1), (1.2) hold.*

We also say that $\mathcal{E} \subset \omega$ is a Pohlke's ellipse for OP_1, OP_2, OP_3 if $\mathcal{E}$ is obtained as above from a Pohlke's projection for OP_1, OP_2, OP_3.

By Pohlke's theorem a Pohlke's projection for OP_1, OP_2, OP_3 always exists and it is unique up to symmetry with respect to ω; the Pohlke's ellipse for OP_1, OP_2, OP_3 is unique.

Notation 1.2 *In the following we will indicate with $\Pi_{\mathtt{P}}$ a Pohlke's projection for OP_1, OP_2, OP_3 and possibly with $\bar{\Pi}_{\mathtt{P}}$ the symmetrical projection with respect to the plane ω. With $\mathcal{E}_{\mathtt{P}}$ we will indicate the Pohlke's ellipse for OP_1, OP_2, OP_3.*

One may note that the existence of an ellipse $\mathcal{E}$ centered at O and circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ does not require a *Pohlke's projection* $\Pi_{\mathtt{P}}$ as above. But merely a parallel projection $\Pi : \mathbb{E}^3 \to \omega$ and equal segments OR_1, OR_2, OR_3 such that $\Pi(OR_i) = OP_i$ $(1 \le i \le 3)$ with

$$\begin{aligned} &i) \quad OR_1 \perp OR_2 \quad \text{and} \quad OR_2 \perp OR_3; \\ &ii) \quad OR_3 \perp OR_1 \quad \text{or} \quad OR_3 \perp OR_1', \end{aligned} \tag{1.3}$$

where the point R_1' is symmetric to R_1 with respect to the plane through O and perpendicular to the direction of the projection. See the Lemma 2.6 below.

Since condition (1.3) is weaker than (1.2), it is natural to ask if there are ellipses $\mathcal{E} \neq \mathcal{E}_{\mathtt{p}}$, with center O, which circumscribe $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. In this paper we will investigate the existence of such ellipses by finding necessary and sufficient conditions for the existence of the corresponding projections $\Pi \neq \Pi_{\mathtt{P}}, \bar{\Pi}_{\mathtt{P}}$ such as to satisfy (1.3) but not (1.2).

To this aim, we give below the relative definitions and then we state the main results.

Definition 1.3 *Let $OP_1, OP_2, OP_3 \subset \omega$ be three segments which are not contained in a line.*

(1) *A parallel projection $\Pi : \mathbb{E}^3 \to \omega$ is a secondary Pohlke's projection for OP_1, OP_2, OP_3 if there are equal segments OR_1, OR_2, OR_3 such that*

$$\Pi(OR_i) = OP_i \quad (1 \le i \le 3), \tag{1.4}$$

$$OR_1 \perp OR_2,\ OR_2 \perp OR_3 \ \text{ and } \ OR_3 \perp OR_1', \tag{1.5}$$

$$R_i \notin \pi \quad (i.e.,\ R_i \neq R_i') \quad (1 \le i \le 3), \tag{1.6}$$

where π is the plane through O and perpendicular to the direction of Π; the point R_i' is symmetric to R_i with respect to π.

(2) *A secondary Pohlke's ellipse for* OP_1, OP_2, OP_3 *is an ellipse* $\mathcal{E} \neq \mathcal{E}_{\mathrm{P}}$*, centered at* O*, which circumscribes the three ellipses* $\mathcal{E}_{P_1,P_2}$*,* $\mathcal{E}_{P_2,P_3}$*,* $\mathcal{E}_{P_3,P_1}$*.* [3]

Theorem 1.4 *Suppose the segments* OP_1*,* OP_2*,* OP_3 *are not contained in a line. Then the following two properties are equivalent:*

(1) *there exists a secondary Pohlke's projection* Π *for* OP_1, OP_2, OP_3*;*

(2) *there exists a secondary Pohlke's ellipse* $\mathcal{E}$ *for* OP_1, OP_2, OP_3*.*

If OP_1, OP_2, OP_3 *are non-parallel, then* (1) *and* (2) *are both equivalent to*

(3) $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$ *with* $h, k \neq 0$ *satisfying*

$$|h| + |k| < 1 \quad or \quad \big||h| - |k|\big| > 1, \tag{1.7}$$

or, equivalently,

$$(h + k + 1)(h + k - 1)(h - k + 1)(h - k - 1) > 0. \tag{1.8}$$

If OP_1, OP_2, OP_3 *are non-parallel and (3) holds, the secondary Pohlke's projection is unique up to symmetry with respect to* ω*; the secondary Pohlke's ellipse is unique and (with reference to Definition 1.3)* $\mathcal{E} = \Pi(S \cap \pi)$ *where* S *is the sphere, with center* O*, containing* R_1, R_2, R_3*.*

Again assuming the segments OP_1, OP_2, OP_3 are not contained in a line, we also have:

Theorem 1.5 *If any two of* OP_1, OP_2, OP_3 *are parallel, then there are infinitely many secondary Pohlke's projections (ellipses) if these two segments are equal, none if they are different.*

Remark 1.6 Condition (1.5) of Definition 1.3 and property (3) of Theorem 1.4 may appear to be non-symmetric with respect to the points P_1, P_2, P_3. However, this is not the case:

(i) Suppose conditions (1.4), (1.5) are verified. Considering also the points R'_2 and R'_3, we can write a cyclic relation of orthogonality:

$$\begin{aligned} &OR_1 \perp OR_2,\ OR_2 \perp OR_3,\ OR_3 \perp OR'_1, \\ &\qquad OR'_1 \perp OR'_2,\ OR'_2 \perp OR'_3,\ OR'_3 \perp OR_1. \end{aligned} \tag{1.9}$$

Hence, possibly replacing some R_i with R'_i and vice versa, from (1.9) it is clear that (1.5) continues to holds for any permutation of the points P_1, P_2, P_3.

(ii) Let us consider property (3) of Theorem 1.4. If $h, k \neq 0$ satisfy one of the conditions of (1.7), it is straightforward to see that one of the following must be verified:

$$|1/h| + |k/h| < 1 \quad \text{or} \quad \big||1/h| - |k/h|\big| > 1. \tag{1.10}$$

Thus, setting $h' = 1/h$ and $k' = -k/h$, we have

$$\overrightarrow{OP_1} = h'\,\overrightarrow{OP_3} + k'\,\overrightarrow{OP_2}, \tag{1.11}$$

[3] If two of segments OP_1, OP_2, OP_3 are parallel (in particular if one of them vanishes) then we need to introduce degenerate ellipses. See Definitions 2.1, 2.2 and 2.4 below.

with $h', k' \neq 0$ such that

$$|h'| + |k'| < 1 \quad \text{or} \quad \big||h'| - |k'|\big| > 1. \tag{1.12}$$

Similarly, we can see that $\overrightarrow{OP_2} = h''\,\overrightarrow{OP_1} + k''\,\overrightarrow{OP_3}$ with $h'' = -h/k$ and $k'' = 1/k$ such that $|h''| + |k''| < 1$ or $\big||h''| - |k''|\big| > 1$. □

Remark 1.7 Condition (1.6) of Definition 1.3 is necessary because if (1.6) fails, then Π is a *Pohlke's projection* for OP_1, OP_2, OP_3; that is, $\Pi = \Pi_{\mathrm{P}}$ or $\Pi = \bar{\Pi}_{\mathrm{P}}$.
Indeed, if OR_1, OR_2, OR_3 are equal segments such that (1.4), (1.5) hold, but $R_j = R'_j$ for some $1 \leq j \leq 3$, then Π satisfies the conditions (1.1), (1.2) just by suitably renaming the points R_i, R'_i $(1 \leq i \leq 3)$. Conversely, if $\Pi = \Pi_{\mathrm{P}}$ or $\Pi = \bar{\Pi}_{\mathrm{P}}$, then one cannot find equal segments OR_1, OR_2, OR_3 such that (1.4), (1.5) and (1.6) hold. See Claim 2.8. □

Remark 1.8 With reference to Definition 1.3, suppose OP_1, OP_2, OP_3 are non-parallel and such that there exists a secondary Pohlke's projection. Then we can obtain the explicit expressions of R_1, R_2, R_3 (as well as the direction of projection) if we know $\angle(OR_1, OR_3)$. To this purpose it is sufficient to apply the same arguments of [5]. □

To conclude we point out that property (3) of Theorem 1.4 is equivalent to the conditions established in [8] for the existence of a secondary *common tangential ellipse*. More precisely, to the conditions given by formula (3.55) of [8, Theorem 3.2] in the circular case (i.e., if one of the three ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ is a circle; see section 3 below) and by the inequalities (3.60), (3.61) of [8, Main Theorem] in the general case.

For instance, let us suppose that OP_1, OP_2, OP_3 are non-parallel and

$$OP_1 \perp OP_2 \quad \text{with} \quad |OP_1| = |OP_2| = \rho > 0. \tag{1.13}$$

Then it is not difficult to see that in (3) of Theorem 1.4 condition (1.7) is verified iff

$$|OP_3| = r\,, \quad \angle(OP_1, OP_3) = \varphi\,,^{4} \tag{1.14}$$

with $r > 0$ and $\varphi \in (0, \widetilde{\varphi}) \cup (\frac{\pi}{2} - \widetilde{\varphi}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{\pi}{2} + \widetilde{\varphi}) \cup (\pi - \widetilde{\varphi}, \pi)$, where

$$\widetilde{\varphi} = \begin{cases} \dfrac{1}{2}\arccos\left(\rho r^{-2}\sqrt{2r^2 - \rho^2}\right) & \text{if} \quad r \geq \dfrac{\rho}{\sqrt{2}}\,, \\ \dfrac{\pi}{2} \qquad \text{if} \quad r < \dfrac{\rho}{\sqrt{2}}\,. \end{cases} \tag{1.15}$$

But (1.15) corresponds to condition (3.55) of [8].

Furthermore, it is readily seen that condition (3.53) of [8], namely

$$1 - 2\left(\frac{r}{\rho}\right)^2 + \left(\frac{r}{\rho}\right)^4 \cos^2 2\varphi > 0\,, \tag{1.16}$$

is equivalent to (1.8). In fact, having (1.13) and $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$, it follows that

$$\left(\frac{r}{\rho}\right)^2 = h^2 + k^2, \quad \cos\varphi = \frac{h}{\sqrt{h^2 + k^2}}\,, \quad \cos 2\varphi = \frac{h^2 - k^2}{h^2 + k^2}\,. \tag{1.17}$$

[4] Here $\angle(OP_1, OP_3)$ is the non-orinted, convex angle between OP_1 and OP_3.

Hence, rewriting (1.16) using the expressions (1.17), we find the condition

$$1 - 2h^2 - 2k^2 + h^4 - 2h^2k^2 + k^4 > 0, \tag{1.18}$$

which coincides with (1.8).

2 Preliminaries

Let ω be a plane in the Euclidean space $\mathbb{E}^3$. Let O be a fixed point of ω.

Definition 2.1 *Suppose $OA, OB \subset \omega$ are two segments such that only one of them can vanish.*

(1) If $OA \nparallel OB$ we denote with $\mathcal{E}_{A,B}$ the ellipse with center O and (OA, OB) as pair of conjugate semi-diameters. We also say that $\mathcal{E}_{A,B}$ is a non-degenerate ellipse.

(2) If $OA \parallel OB$ (in particular if one of them vanishes) we denote with $\mathcal{E}_{A,B}$ the straight line segment $VW \parallel OA, OB$ with O as midpoint and such that

$$|VW| = 2\sqrt{|OA|^2 + |OB|^2}\,. \tag{2.1}$$

We say that $\mathcal{E}_{A,B}$ is the degenerate ellipse given by the pair of conjugate, parallel semi-diameters (OA, OB). [5]

For $\rho > 0$ we indicate with $S(\rho)$ the sphere with center O and radius ρ:

$$S(\rho) = \{P \in \mathbb{E}^3 : |OP| = \rho\}. \tag{2.2}$$

Definition 2.2 *Let $\mathcal{E} \subset \omega$ be a non-degenerate ellipse with center O.*

(1) If $\mathcal{E}$ has semi-minor axis $b > 0$, for brevity we write $S_{\mathcal{E}} = S(b)$.

(2) We indicate with $\widetilde{\mathcal{E}}$ the set of points $P \in \omega$ enclosed by $\mathcal{E}$. More precisely, if $\mathcal{E}$ has foci F_1, F_2 and semi-major axis $a > 0$, then

$$\widetilde{\mathcal{E}} = \{P \in \omega : |F_1P| + |F_2P| \le 2a\}. \tag{2.3}$$

(3) We indicate with $\Pi_{\mathcal{E}} : \mathbb{E}^3 \to \omega$ a parallel projection onto ω such that $\Pi_{\mathcal{E}}(S_{\mathcal{E}}) = \widetilde{\mathcal{E}}$. If $\Omega \subset \widetilde{\mathcal{E}}$, for simplicity we set

$$\widetilde{\Pi}_{\mathcal{E}}^{-1}(\Omega) = \big\{P \in \mathbb{E}^3 \;:\; \Pi_{\mathcal{E}}(P) \in \Omega\big\} \cap S_{\mathcal{E}}. \tag{2.4}$$

(4) We denote with $\pi_{\mathcal{E}}$ the plane through O perpendicular to the direction of the projection $\Pi_{\mathcal{E}}$ and with $\mathcal{C}_{\mathcal{E}}$ the great circle of $S_{\mathcal{E}}$ given by $\mathcal{C}_{\mathcal{E}} = S_{\mathcal{E}} \cap \pi_{\mathcal{E}}$.

Remark 2.3 We note, without proof, the following elementary facts:

[5] See also [1], pp. 372-373.

(a) Given a non-degenerate ellipse $\mathcal{E} \subset \omega$ with center O, there are (in general) two oblique projections, say $\Pi_{\mathcal{E}}$ and $\bar{\Pi}_{\mathcal{E}}$, such that

$$\Pi_{\mathcal{E}}(S_{\mathcal{E}}) = \bar{\Pi}_{\mathcal{E}}(S_{\mathcal{E}}) = \widetilde{\mathcal{E}}.$$

These projections are symmetric with respect to ω in the sense that if the points $P, \bar{P}$ are symmetric with respect to ω, then $\Pi_{\mathcal{E}}(P) = \bar{\Pi}_{\mathcal{E}}(\bar{P})$. Further, $\Pi_{\mathcal{E}} = \bar{\Pi}_{\mathcal{E}}$ iff $\mathcal{E}$ is a circle; in this case $\Pi_{\mathcal{E}}$ is the orthogonal projection onto ω. So we may say that $\Pi_{\mathcal{E}}$ is unique up to symmetry with respect to ω. For this reason in the following we will not distinguish between $\Pi_{\mathcal{E}}$ and $\bar{\Pi}_{\mathcal{E}}$. We will limit ourselves to choose on of them, since the meaning will always be clear from the context.

(b) Let $\Pi : \mathbb{E}^3 \to \omega$ be a parallel projection and let π be the plane through O and perpendicular to the direction of Π. If S is a sphere with center O, then

$$\mathcal{E} = \Pi(S \cap \pi)$$

is a non-degenerate ellipse, centered at O, such that $S_{\mathcal{E}} = S$ and

$$\Pi = \Pi_{\mathcal{E}} \quad \text{or} \quad \Pi = \bar{\Pi}_{\mathcal{E}}. \tag{2.5}$$

(c) Let $\mathcal{E}_1, \mathcal{E}_2 \subset \omega$ be two non-degenerate ellipses with center O. Then $\mathcal{E}_1, \mathcal{E}_2$ are homothetic with respect to the center O iff $\Pi_{\mathcal{E}_1} = \Pi_{\mathcal{E}_2}$ or $\Pi_{\mathcal{E}_1} = \bar{\Pi}_{\mathcal{E}_2}$.

(d) Given $P \in \widetilde{\mathcal{E}}$, we have

$$\widetilde{\Pi}_{\mathcal{E}}^{-1}(P) = \{Q, Q'\} \subset S_{\mathcal{E}}, \tag{2.6}$$

with Q, Q' symmetric with respect to $\pi_{\mathcal{E}}$; moreover, $Q = Q'$ iff $P \in \mathcal{E}$.
We note also that $\mathcal{C}_{\mathcal{E}} = \widetilde{\Pi}_{\mathcal{E}}^{-1}(\mathcal{E})$ and $\Pi_{\mathcal{E}}(\mathcal{C}_{\mathcal{E}}) = \mathcal{E}$.

Definition 2.4 *Let $\mathcal{E}_1, \mathcal{E}_2 \subset \omega$ be two given ellipses with center O and let $\mathcal{E}_1$ be non-degenerate.*

(1) If $\mathcal{E}_2$ is non-degenerate, we say that $\mathcal{E}_1$ circumscribes $\mathcal{E}_2$ if $\mathcal{E}_2 \subset \widetilde{\mathcal{E}_1}$ and $\mathcal{E}_1, \mathcal{E}_2$ are tangent at two points S, T which are symmetric with respect to O.

(2) If $\mathcal{E}_2 = VW$ is degenerate, we say that $\mathcal{E}_1$ circumscribes $\mathcal{E}_2$ if $V, W \in \mathcal{E}_1$. In this case we also say that $\mathcal{E}_2$ is tangent to $\mathcal{E}_1$ at the points V, W.

Before continuing we recall (see [7]) that affine transformations map pairs of conjugate semi-diameters of a central conic into pairs of conjugate semi-diameters of the transformed conic (for degenerate ellipses, in the sense of Definition 2.1, this is obvious). In the case of a circle conjugate semi-diameters are perpendicular. We also remark that non-degenerate conics with a given center O are uniquely determined by 3 (independent and compatible) conditions. For instance if we know that a conic passes through two distinct points A, B and has tangent t at one of them (provided $OA \nparallel OB$, $AB \nparallel t$ and $O \notin t$).

If $\Pi : \mathbb{E}^3 \to \omega$ is a parallel projection onto the plane ω and if $\omega_1 \subset \mathbb{E}^3$ is any plane not parallel to the direction of Π, then the restriction

$$\Pi\Big|_{\omega_1} : \omega_1 \to \omega$$

defines an invertible affine map between ω_1 and ω.

Taking into account these facts, we notice the following elementary consequences:

Claim 2.5 Let $\mathcal{E} \subset \omega$ be a non-degenerate ellipse with center O and let $\Pi_{\mathcal{E}} : \mathbb{E}^3 \to \omega$ be a projection as in (3) of Definition 2.2. Suppose further that $\mathcal{E}$ circumscribes $\mathcal{E}_{A,B}$.

(a) If $\mathcal{E}_{A,B} = \mathcal{E}$, then $\widetilde{\Pi}_{\mathcal{E}}^{-1}(A) = C$, $\widetilde{\Pi}_{\mathcal{E}}^{-1}(B) = D$ with $C, D \in \mathcal{C}_{\mathcal{E}}$ such that $OC \perp OD$.

(b) If $\mathcal{E}_{A,B} \neq \mathcal{E}$ and $\mathcal{E}_{A,B}$ is non-degenerate then $\widetilde{\Pi}_{\mathcal{E}}^{-1}(\mathcal{E}_{A,B}) = \mathcal{C} \cup \mathcal{C}'$ where $\mathcal{C}$, $\mathcal{C}'$ are two distinct great circles of $S_{\mathcal{E}}$, which are symmetric with respect to the plane $\pi_{\mathcal{E}}$. There exist $C, D \in \mathcal{C}$ and $C', D' \in \mathcal{C}'$ such that $\{C, C'\} = \widetilde{\Pi}_{\mathcal{E}}^{-1}(A)$, $\{D, D'\} = \widetilde{\Pi}_{\mathcal{E}}^{-1}(B)$ and

$$OC \perp OD, \quad OC' \perp OD'. \tag{2.7}$$

(c) If $\mathcal{E}_{A,B} = VW$ is degenerate, $\widetilde{\Pi}_{\mathcal{E}}^{-1}(\mathcal{E}_{A,B}) = \mathcal{C}$ is the great circle of $S_{\mathcal{E}}$ in the plane through VW and parallel to the direction of projection. Then, by condition (2.1), we can still select $C, C', D, D' \in \mathcal{C}$ such that $\{C, C'\} = \widetilde{\Pi}_{\mathcal{E}}^{-1}(A)$, $\{D, D'\} = \widetilde{\Pi}_{\mathcal{E}}^{-1}(B)$ and (2.7) holds.

Proof of (a). By hypothesis OA, OB are conjugate semi-diameters of $\mathcal{E}$. This implies that OC and OD are conjugate semi-diameters of the circle $\mathcal{C}_{\mathcal{E}}$. Hence they are perpendicular. □

Proof of (b). In this case we have $\{A, B\} \not\subset \mathcal{E}$ and $OA \nparallel OB$. Let us suppose, for instance, that $A \notin \mathcal{E}$. Then $\widetilde{\Pi}_{\mathcal{E}}^{-1}(A) = \{C, C'\}$ with $C \neq C'$ and the ellipses $\mathcal{E}$, $\mathcal{E}_{A,B}$ are tangent at a point $P \neq A$. Thus $OA \nparallel OP$.
Setting $Q = \widetilde{\Pi}_{\mathcal{E}}^{-1}(P)$, we define $\mathcal{C}$ and $\mathcal{C}'$ as the great circles of $S_{\mathcal{E}}$ through C, Q and C', Q respectively. This definition gives $\mathcal{C} \neq \mathcal{C}'$. Indeed, if $\mathcal{C} = \mathcal{C}'$, then C belongs to the plane through OQ and parallel to the direction of projection (i.e., CC'). But then $OA = \Pi(OC) \parallel \Pi(OQ) = OP$, contrary to our assumption. Besides, $\mathcal{C}$ and $\mathcal{C}'$ are symmetric with respect to $\pi_{\mathcal{E}}$ because $Q \in \pi_{\mathcal{E}}$ and C, C' are symmetric with respect to $\pi_{\mathcal{E}}$. Next, we observe that

$$\Pi_{\mathcal{E}}(\mathcal{C}) = \Pi_{\mathcal{E}}(\mathcal{C}') = \mathcal{E}_{A,B}\,, \tag{2.8}$$

because the three non-degenerate ellipses $\Pi_{\mathcal{E}}(\mathcal{C})$, $\Pi_{\mathcal{E}}(\mathcal{C}')$ and $\mathcal{E}_{A,B}$ have the same center, they pass through the point A, and they are tangent at the point P. Hence we can select $D \in \mathcal{C}$, $D' \in \mathcal{C}'$ such that $\{D, D'\} = \widetilde{\Pi}_{\mathcal{E}}^{-1}(B)$. Then (2.7) holds, because (OC, OD) and (OC', OD') are pairs of conjugate semi-diameters for $\mathcal{C}$, $\mathcal{C}'$ respectively. □

Proof of (c). Let us suppose first $|OA|, |OB| \neq 0$, that is $A, B \notin \mathcal{E}$. Then we have $\widetilde{\Pi}_{\mathcal{E}}^{-1}(A) = \{C, C'\}$, $\widetilde{\Pi}_{\mathcal{E}}^{-1}(B) = \{D, D'\}$ with $C \neq C'$ and $D \neq D'$. Setting $X = \widetilde{\Pi}_{\mathcal{E}}^{-1}(V)$ and $Y = \widetilde{\Pi}_{\mathcal{E}}^{-1}(W)$, we easily see that XY is the diameter of $\mathcal{C}$ orthogonal to the direction of projection. Hence $XY \perp CC', DD'$ and, by condition (2.1), the points

$$E = CC' \cap XY \quad \text{and} \quad F = DD' \cap XY$$

are such that

$$|XY| = 2\sqrt{|OE|^2 + |OF|^2}\,. \tag{2.9}$$

Now, since EC, $FD \perp XY$, from (2.9) we easily deduce that

$$|OE| = |FD|\,, \quad |OF| = |EC| \quad \text{and then} \quad \overrightarrow{OE} \cdot \overrightarrow{OF} = \pm\, \overrightarrow{EC} \cdot \overrightarrow{FD}. \tag{2.10}$$

Taking into account that C, C' are symmetric with respect to $\pi_{\mathcal{E}}$, we find

$$\overrightarrow{OC} \cdot \overrightarrow{OD} = \left(\overrightarrow{OE} + \overrightarrow{EC}\right) \cdot \left(\overrightarrow{OF} + \overrightarrow{FD}\right) = \overrightarrow{OE} \cdot \overrightarrow{OF} + \overrightarrow{EC} \cdot \overrightarrow{FD}, \tag{2.11}$$

$$\overrightarrow{OC'} \cdot \overrightarrow{OD} = \left(\overrightarrow{OE} - \overrightarrow{EC}\right) \cdot \left(\overrightarrow{OF} + \overrightarrow{FD}\right) = \overrightarrow{OE} \cdot \overrightarrow{OF} - \overrightarrow{EC} \cdot \overrightarrow{FD}. \tag{2.12}$$

By the last of (2.10) we have $OC \perp OD$ or $OC' \perp OD$ and, by symmetry, $OC' \perp OD'$ or $OC \perp OD'$. Hence, possibly renaming the points C, C' and D, D', we can verify (2.7).

Finally, let us suppose $|OA| = 0$ (if $|OB| = 0$ the proof is the same). We have $D = D' \in \mathcal{C}_{\mathcal{E}}$ because $B \in \mathcal{E}$. Further, CC' is the diameter of $\mathcal{C}$ parallel to the direction of projection. It follows that $OC \perp OD$ and $OC' \perp OD$. Thus (2.7) holds true with $D' = D$. □

From the previous definitions and Claim 2.5, we have:

Lemma 2.6 *Let $OP_1, OP_2, OP_3 \subset \omega$ be three arbitrary segments which are not contained in a line. Then the following hold:*

(1) *Let $\Pi : \mathbb{E}^3 \to \omega$ be a parallel projection onto ω and let OQ_1, OQ_2, OQ_3 be equal line segments such that:*

(i) $\Pi(OQ_i) = OP_i \quad (1 \le i \le 3)$

(ii) $OQ_1 \perp OQ_2\,, \quad OQ_2 \perp OQ_3$

and

(iii) $OQ_3 \perp OQ_1$ *or* *(iv)* $OQ_3 \perp OQ_1'$

where, in case (iv), Q_1' is symmetric to Q_1 with respect to the plane π passing through O and perpendicular to the direction of the projection. Then, denoting with S the sphere with center O and containing the points Q_i $(1 \le i \le 3)$, $\mathcal{E} = \Pi(S \cap \pi)$ is a non-degenerate ellipse circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$.

(2) *Conversely, let $\mathcal{E}$ be a non-degenerate ellipse with center O and circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$. Then, setting*

$$\Pi = \Pi_{\mathcal{E}}\,,$$

we can find $Q_1, Q_2, Q_3 \in S_{\mathcal{E}}$ such that the segments OQ_1, OQ_2, OQ_3 verify the conditions (i),(ii) and (iii) or (iv) of (1).

Proof of (1). Let us suppose, for instance, that Π and OQ_1, OQ_2, OQ_3 satisfy the conditions (i), (ii) and (iv) of (1). To begin with, we set:

$$\mathcal{C} = S \cap \pi \quad \text{and} \quad \mathcal{E} = \Pi(\mathcal{C}). \tag{2.13}$$

Being the parallel projection of the great circle $\mathcal{C} \subset \pi$, $\mathcal{E}$ is a non-degenerate ellipse with center O. Next, we consider the planes $\pi_{1,2}$, $\pi_{2,3}$ and $\pi_{3,1}'$ passing through O and the couples of points (Q_1, Q_2), (Q_2, Q_3) and (Q_3, Q_1') respectively. Let

$$\mathcal{C}_{1,2} = S \cap \pi_{1,2}, \quad \mathcal{C}_{2,3} = S \cap \pi_{2,3} \quad \text{and} \quad \mathcal{C}_{3,1}' = S \cap \pi_{3,1}' \tag{2.14}$$

be the corresponding great circles of S. By assumption (i) of (1) and since

$$\Pi(Q_1') = \Pi(Q_1), \tag{2.15}$$

it is clear that $\Pi(\mathcal{C}_{1,2})$, $\Pi(\mathcal{C}_{2,3})$, $\Pi(\mathcal{C}_{3,1}')$ are three (possibly degenerate) ellipses with center O, passing through the couples of points (P_1, P_2), (P_2, P_3) and (P_3, P_1) respectively.

Moreover, by conditions (ii) and (iv), (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1) are (even in the degenerate case) pairs of conjugate semi-diameters for $\Pi(\mathcal{C}_{1,2})$, $\Pi(\mathcal{C}_{2,3})$, $\Pi(\mathcal{C}'_{3,1})$ respectively. Hence we deduce that

$$\Pi(\mathcal{C}_{1,2}) = \mathcal{E}_{P_1,P_2}, \quad \Pi(\mathcal{C}_{2,3}) = \mathcal{E}_{P_2,P_3}, \quad \Pi(\mathcal{C}'_{3,1}) = \mathcal{E}_{P_3,P_1}. \tag{2.16}$$

It follows that $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1} \subset \widetilde{\mathcal{E}} = \Pi(S)$ and that $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ are tangent to $\mathcal{E}$ since the great circles $\mathcal{C}_{1,2}$, $\mathcal{C}_{2,1}$, $\mathcal{C}'_{3,1}$ must intersect $\mathcal{C}$. Thus we have proved that $\mathcal{E}$ circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

If Π and OQ_1, OQ_2, OQ_3 verify conditions (i), (ii) and (iii) of (1) the proof is similar. □

Proof of (2). Let us suppose that the assumptions of (2) apply. Setting $\Pi = \Pi_{\mathcal{E}}$, we must show that there exist OQ_1, OQ_2, OQ_3 which satisfy the hypothesis of (1) of Lemma 2.6.

Starting with $\mathcal{E}_{P_1,P_2}$ and using the appropriate statement (i.e., (a), (b) or (c)) of Claim 2.5, we can find $Q_1, Q_2 \in S_{\mathcal{E}}$ such that

$$\Pi(Q_1) = P_1,\ \Pi(Q_2) = P_2 \quad \text{and} \quad OQ_1 \perp OQ_2\,.$$

Then we consider $\mathcal{E}_{P_2,P_3}$. Using Claim 2.5 once again, we find $Q_3 \in S_{\mathcal{E}}$ such that

$$\Pi(Q_3) = P_3 \quad \text{and} \quad OQ_2 \perp OQ_3\,.$$

In this way the conditions $(i), (ii)$ of part (1) are certainly verified.

Finally, let us consider $\mathcal{E}_{P_3,P_1}$. In this cases, since we have already choose $Q_1 \in \widetilde{\Pi}_{\mathcal{E}}^{-1}(P_1)$ and $Q_3 \in \widetilde{\Pi}_{\mathcal{E}}^{-1}(P_3)$, by Claim 2.5 we can only say that

$$Q_3 \perp Q_1 \quad \text{or} \quad Q_3 \perp Q'_1, \tag{2.17}$$

where Q'_1 is symmetric to Q_1 with respect to the plane $\pi_{\mathcal{E}}$.

Hence at least one of the two conditions (iii), (iv) of (1) must be verified. □

Remark 2.7 In proving (2) of Lemma 2.6 it is worthwhile to note the following:

(a) If $P_i \in \mathcal{E}$ for some $1 \leq i \leq 3$, then both cases of (1) can be verified. More precisely, if Q_1, Q_2, Q_3 are such that (i), (ii), (iii) hold, then possibly after exchanging Q_j with Q'_j for some $1 \leq j \leq 3$, we obtain (i), (ii), (iv). Similarly, if (i), (ii), (iv) hold, after exchanging Q_j with Q'_j for some $1 \leq j \leq 3$, we get (i), (ii), (iii).
For instance, let us suppose that (i), (ii), (iii) hold and that $P_2 \in \mathcal{E}$. Then, since $Q_2 = Q'_2$, by symmetry with respect to the plane $\pi_{\mathcal{E}}$ we can write:

$$OQ_1 \perp OQ_2\,, \quad OQ_2 \perp OQ'_3\,, \quad OQ'_3 \perp OQ'_1. \tag{2.18}$$

Thus, by exchanging Q_3 with Q'_3, we have (i), (ii), (iv).

(b) Conversely, if both cases of (1) can be verified, up to exchanging Q_j with Q'_j for some $1 \leq j \leq 3$, then $Q_i = Q'_i$ and $P_i \in \mathcal{E}$ for some $1 \leq i \leq 3$.

In fact, suppose the points Q_1, Q_2, Q_3 are such that both cases of (1) can be verified up to exchanging Q_j with Q'_j for some $1 \leq j \leq 3$. In this situation we can easily see that

$$OQ_h \perp OQ_k \quad \text{and} \quad OQ_h \perp OQ'_k, \tag{2.19}$$

for some $h \neq k$, with $1 \leq h, k \leq 3$. [6]
Now, if $Q_k = Q'_k$, then $Q_k \in \pi_{\mathcal{E}}$ and $P_k \in \mathcal{E}$. Otherwise, $Q_k Q'_k$ is a nonzero segment parallel to the direction of projection and from (2.19) it follows that

$$OQ_h \perp Q_k Q'_k. \tag{2.20}$$

By definition, this means that $Q_h \in \pi_{\mathcal{E}}$. Thus $Q_h = Q'_h$ and $P_h \in \mathcal{E}$. □

Summing up, we have showed that in proving (2) of Lemma 2.6 *one can verify both the cases of* (1) (*that is* $(i), (ii), (iii)$ *or* $(i), (ii), (iv)$) *possibly up to exchanging some* Q_j *with* Q'_j $\Leftrightarrow$ *at least one of the points* P_1, P_2, P_3 *belongs to* $\mathcal{E}$.

Continuing to assume that OP_1, OP_2, OP_3 are not contained in a line, we conclude this section by showing that if a projection $\Pi : \mathbb{E}^3 \to \omega$ satisfies the conditions of (1) of Lemma 2.6, then Π is either a Pohlke's projection or a secondary Pohlke's projection for OP_1, OP_2, OP_3.

Claim 2.8 Let $\Pi : \mathbb{E}^3 \to \omega$ be parallel projection. Suppose there exist equal line segments OR_1, OR_2, OR_3 such that:

$$\Pi(OR_i) = OP_i \quad (1 \leq i \leq 3),$$

$$OR_1 \perp OR_2, \quad OR_2 \perp OR_3 \quad \text{and} \quad OR_3 \perp OR'_1.$$

Then the following facts are equivalent:

(a) Π is a Pohlke's projection for OP_1, OP_2, OP_3;

(b) $R_i \in \pi$ (i.e., $R_i = R'_i$) for some $1 \leq i \leq 3$.

Proof (b) $\Rightarrow$ (a). If $R_i = R'_i$ for some $1 \leq i \leq 3$ (where R'_i is symmetric to R_i with respect to the plane π) we can easily see that Π satisfies the conditions (1.1), (1.2) for a suitable choice of $Q_j \in \{R_j, R'_j\}$ $(1 \leq j \leq 3)$. For instance, if $R_2 = R'_2$, we can choose:

$$Q_1 = R'_1, \quad Q_2 = R_2, \quad Q_3 = R_3. \tag{2.21}$$

The other cases are similar. Thus $\Pi = \Pi_{\mathrm{P}}$ or $\Pi = \bar{\Pi}_{\mathrm{P}}$.
Proof (a) $\Rightarrow$ (b). Conversely, assume that $\Pi = \Pi_{\mathrm{P}}$ or $\Pi = \bar{\Pi}_{\mathrm{P}}$. Then there exist equal segments OQ_1, OQ_2, OQ_3 such that conditions (1.1), (1.2) hold.
We may suppose that $Q_i \in S$ and $R_i \in \tilde{S}$ $(1 \leq i \leq 3)$ where S and $\tilde{S}$ are suitable spheres with center O. By (1) of Lemma 2.6,

$$\mathcal{E} = \Pi(S \cap \pi) \quad \text{and} \quad \tilde{\mathcal{E}} = \Pi(\tilde{S} \cap \pi) \tag{2.22}$$

are two ellipses circumscribing $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$. Since $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are homothetic with respect to the point O, we deduce that $\mathcal{E} = \tilde{\mathcal{E}}$. Thus $S = \tilde{S}$ and

$$\{R_i, R'_i\} = \{Q_i, Q'_i\} \quad (1 \leq i \leq 3). \tag{2.23}$$

[6] Indeed, let $\{R_i, R'_i\} = \{Q_i, Q'_i\}$ for $1 \leq i \leq 3$. Also, assume that both $Q_1 \perp Q_2$, $Q_2 \perp Q_3$, $Q_3 \perp Q_1$ and $R_1 \perp R_2$, $R_2 \perp R_3$, $R_3 \perp R'_1$ are verified. Then, rewriting the second expression in terms of Q_i and Q'_i, we can see that (2.19) holds for any feasible choice of R_i, R'_i.

This means that the points Q_1, Q_2, Q_3 are such that both the cases (i), (ii), (iii) and (i), (ii), (iv) of (1) of Lemma 2.6 can be verified, possibly up to exchanging Q_j with Q'_j for some $1 \le j \le 3$. Hence, for some $h \neq k$ $(1 \le h, k \le 3)$, we must have

$$OQ_h \perp OQ_k \quad \text{and} \quad OQ_h \perp OQ'_k, \tag{2.24}$$

as we have already seen in (b) of Remark 2.7. This implies that $Q_i \in \pi$ for some $1 \le i \le 3$. Thus, by (2.23), $R_i \in \pi$ for some $1 \le i \le 3$. □

Next, repeatedly applying the implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) of Claim 2.8 and taking into account Definitions 1.1 and 1.3, we can prove that:

Lemma 2.9 *If $\Pi : \mathbb{E}^3 \to \omega$ satisfies the conditions of (1) of Lemma 2.6, then Π is either a Pohlke's projection or a secondary Pohlke's projection (but not both) for OP_1, OP_2, OP_3. In particular, a secondary Pohlke's projection cannot be also a Pohlke's projection.*

Proof. We distinguish three cases:

If Π verifies the conditions (i), (ii) and (iii) of (1) of Lemma 2.6, then $\Pi = \Pi_{\mathsf{P}}$ or $\Pi = \bar{\Pi}_{\mathsf{P}}$. Moreover, by the implication (a) $\Rightarrow$ (b) of Claim 2.8, Π is not a secondary Pohlke's projection for OP_1, OP_2, OP_3 because if OR_1, OR_2, OR_3 are equal segments such that (1.4), (1.5) hold, then condition (1.6) cannot be verified.

Next, let us suppose that Π verifies the conditions (i), (ii) and (iv) of (1) of Lemma 2.6.

If $Q_i \in \pi$ for some $1 \le i \le 3$, then $\Pi = \Pi_{\mathsf{P}}$ or $\Pi = \bar{\Pi}_{\mathsf{P}}$ by the implication (b) $\Rightarrow$ (a) of Claim 2.8. Further, using the implication (a) $\Rightarrow$ (b) of Claim 2.8 as in the previous case, we deduce that Π is not a secondary Pohlke's projection for OP_1, OP_2, OP_3.

If $Q_i \notin \pi$ for $1 \le i \le 3$, then Π is clearly a secondary Pohlke's projection for OP_1, OP_2, OP_3. Furthermore, once again from Claim 2.8, applying the contrapositive implication $\neg$ (b) $\Rightarrow$ $\neg$ (a), we deduce that Π cannot be a Pohlke's projection for OP_1, OP_2, OP_3. □

3 Pohlke's type projections in the circular case

Let $OP_1, OP_2, OP_3 \subset \omega$ be three segments which are not contained in a line.

We consider now the problem of finding Pohlke's and secondary Pohlke's projections for OP_1, OP_2, OP_3 assuming that

$$OP_1 \perp OP_2 \quad \text{and} \quad |OP_1| = |OP_2| = \rho > 0. \tag{3.1}$$

That is, $\mathcal{E}_{P_1,P_2}$ is a circle with center O and radius ρ.

In this case it is clear that an ellipse $\mathcal{E}$ with center O and circumscribing $\mathcal{E}_{P_1,P_2}$ must have semi-minor axis $b = \rho$. Hence we have

$$S_{\mathcal{E}} = S(\rho). \tag{3.2}$$

Pohlke's projection

We already know that there always exists a Pohlke's projection Π_{P}, which is unique up to symmetry with respect to ω. In view of (3.1), (3.2) it is easy to find it explicitly.

Indeed, since $P_1, P_2 \in S_{\mathcal{E}}$, we must have $P_1 = Q_1$ or Q_1' and $P_2 = Q_2$ or Q_2'. Having $OP_1 \perp OP_2$, it is not restrictive to define:

$$Q_1 = P_1\,, \quad Q_2 = P_2 \tag{3.3}$$

(or, equivalently, $Q_1' = P_1$ and $Q_2' = P_2$). [7]

The Pohlke's projection for OP_1, OP_2, OP_3 is then determined by the choice of $Q_3 \in S(\rho)$ such that $\Pi(Q_3) = P_3$ and $OQ_3 \perp OQ_1, OQ_2$. It turns out that we must take

$$\overrightarrow{OQ_3} = \pm \frac{1}{\rho}\, \overrightarrow{OP_1} \wedge \overrightarrow{OP_2}, \tag{3.4}$$

and that the direction of projection is given by the nonzero vector $\overrightarrow{Q_3P_3}$, because $Q_3 \notin \omega$.

Choosing the plus and then the minus sign in (3.4), we obtain two Pohlke's projections Π_{P}, $\bar{\Pi}_{\mathrm{P}}$ (according to Notation 1.2) which are clearly symmetric with respect to the plane ω.

We may conclude that there is unique Pohlke's projection, up to symmetry with respect to ω, and that there are no restrictions on the segment $OP_3 \subset \omega$.

Secondary Pohlke's projection

Still assuming (3.1), we shall see that there exists a secondary Pohlke's projection if and only if $OP_3 \subset \omega$ satisfies suitable conditions. To begin with, for secondary Pohlke's projection it is necessary to set

$$R_1 = P_1\,, \quad R_2 = P_2 \tag{3.5}$$

(or, equivalently, $R_1' = P_1$ and $R_2' = P_2$). [8]

Having fixed R_1, R_2 as in (3.5) and taking into account Definition 1.3, we need to find $R_3, R_3' \in S(\rho)$ such that:

(1) $OP_2 \perp OR_3$ and $OP_1 \perp OR_3'$ (i.e., $OR_3 \perp OP_1'$);

(2) $R_3 \neq R_3'$ (i.e., $R_3 \notin \pi$);

(3) $R_3R_3' \not\perp OP_1$ and $R_3R_3' \not\perp OP_2$, because we require that $R_1, R_2 \notin \pi$;

(4) $R_3R_3' \not\parallel \omega$, because R_3R_3' gives the direction of projection onto ω;

(5) R_3, R_3', P_3 are collinear (i.e., $\Pi(R_3) = \Pi(R_3') = P_3$).

[7] If, for instance, we try to define $Q_1 = P_1$ and $Q_2' = P_2$ and if this choice works, then it follows that $OQ_1 \perp OQ_2$ and $OQ_1 \perp OQ_2'$. Hence $Q_1 = Q_1'$ or $Q_2 = Q_2'$, by the same argument used in (b) of Remark 2.7. This means that we have $Q_1' = P_1$, $Q_2' = P_2$ in the first case and $Q_1 = P_1$, $Q_2 = P_2$ in the second one. In conclusion, the choice $Q_1 = P_1$, $Q_2' = P_2$ (when it works) is equivalent to (3.3).

[8] In fact, if we try to define $R_1 = P_1$ and $R_2' = P_2$ (or, equivalently, $R_1' = P_1$ and $R_2 = P_2$), then we have $OR_1 \perp OR_2$ and $OR_1 \perp OR_2'$. Hence $R_1 = R_1'$ or $R_2 = R_2'$, by the same argument used in (b) of Remark 2.7. Thus Π cannot be a secondary Pohlke's projection.

Remark 3.1 We may observe that:

$$(1) \wedge (2) \wedge (3) \quad \Rightarrow \quad \overrightarrow{OR_3},\, \overrightarrow{OR_3'} \nparallel \overrightarrow{OP_1} \wedge \overrightarrow{OP_2}\,.\ ^{9} \tag{3.6}$$

In view of (3.4), this immediately implies that $\Pi \neq \Pi_{\mathbf{P}},\, \bar{\Pi}_{\mathbf{P}}$. □

To proceed further, we introduce a cartesian system of coordinate axes x, y, z oriented in space and scaled such that ω is the plane $z = 0$,

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad P_3 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \tag{3.7}$$

In particular, in this system we have

$$\overrightarrow{OP_3} = x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}\,. \tag{3.8}$$

Then the conditions from (1) to (4) are satisfied iff R_3, R_3' are of the form

$$R_3 = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} \quad \text{and} \quad R_3' = \begin{pmatrix} 0 \\ \cos\beta \\ \sin\beta \end{pmatrix}, \tag{3.9}$$

with α, β such that

$$\cos\alpha,\, \cos\beta \neq 0 \quad \text{and} \quad \sin\alpha \neq \sin\beta, \tag{3.10}$$

while taking into account (3.9), condition (5) holds iff

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} + t \begin{pmatrix} -\cos\alpha \\ \cos\beta \\ \sin\beta - \sin\alpha \end{pmatrix} \quad \text{for some} \quad t \in \mathbb{R}. \tag{3.11}$$

Now, assuming that (3.10) holds, we will study the solvability of the system (3.11). We will distinguish three cases to this aim:

<u>**Case** $x = 0$.</u> Since $\cos\alpha \neq 0$, the first equation of (3.11) gives $t = 1$. Then, considering also the third equation, we have $\sin\beta = 0$. Thus, $\cos\beta = \pm 1$ and $\sin\alpha \neq 0$. Summarizing up, when $x = 0$ system (3.11) is solvable iff

$$P_3 = \pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.12}$$

If (3.12) holds, then we have

$$R_3 = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} \quad \text{and} \quad R_3' = P_3, \tag{3.13}$$

[9] If $OR_3 \parallel \overrightarrow{OP_1} \wedge \overrightarrow{OP_2}$, then $OP_1 \perp OR_3$. By the conditions (1) and (2) it follows that $R_3R_3' \perp OP_1$, contrary to (3). If $OR_3' \parallel \overrightarrow{OP_1} \wedge \overrightarrow{OP_2}$ we can argue similarly.

with α such that $\cos\alpha \neq 0, \pm 1$. This means that there are no secondary Pohlke's projections if (3.12) fails, and infinitely many if (3.12) holds. □

Case $y = 0$. Similar reasoning leads to the result that when $y = 0$ (3.11) is solvable iff

$$P_3 = \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.14}$$

If (3.14) holds, then we have

$$R_3 = P_3 \quad \text{and} \quad R_3' = \begin{pmatrix} 0 \\ \cos\beta \\ \sin\beta \end{pmatrix}, \tag{3.15}$$

with β such that $\cos\beta \neq 0, \pm 1$. Hence there are no secondary Pohlke's projections if (3.14) fails, and infinitely many if (3.14) holds. □

Summing up the previous cases:

Lemma 3.2 *If condition (3.1) is verified and if $OP_3 \parallel OP_1$ (or $\parallel OP_2$) then there are infinitely many secondary Pohlke's projection for OP_1, OP_2, OP_3 if $|OP_3| = |OP_1|$ (or $|OP_2|$), none if $|OP_3| \neq |OP_1|$ (or $|OP_2|$).*

Case $x, y \neq 0$. If $x, y \neq 0$ we have an additional condition on α, β. Namely,

$$\sin\alpha,\ \sin\beta \neq 0. \tag{3.16}$$

Indeed, if $\sin\alpha = 0$, (3.10) and the third equation of (3.11) give $t = 0$. Then, the second equation of (3.11) implies $y = 0$, contrary to our assumption. Similarly we find that $\sin\beta \neq 0$.

Taking into account this fact, we will deduce a set of necessary conditions for a point $P_3 = {}^t(x, y, 0)$ to be collinear with R_3, R_3' (i.e., to satisfy (3.11) for some $t \in \mathbb{R}$) when (3.10) and (3.16) are verified. After that, we will prove that these conditions are also sufficient.

To begin with, by (3.10) and the third equation of (3.11), we have

$$t = \frac{\sin\alpha}{\sin\alpha - \sin\beta}. \tag{3.17}$$

Applying (3.16) it follows that $t \neq 0, 1$ and that

$$x = \cos\alpha - \frac{\cos\alpha\sin\alpha}{\sin\alpha - \sin\beta} \quad \Rightarrow \quad x \neq 0,\ \cos\alpha; \tag{3.18}$$

$$y = \frac{\cos\beta\sin\alpha}{\sin\alpha - \sin\beta} \quad \Rightarrow \quad y \neq 0,\ \cos\beta. \tag{3.19}$$

Then

$$\frac{x}{\cos\alpha} + \frac{y}{\cos\beta} = 1. \tag{3.20}$$

From (3.19), (3.20) we obtain

$$\begin{aligned} \cos\alpha &= \frac{x\cos\beta}{\cos\beta - y}\,, \\ \sin\alpha &= \frac{y\sin\beta}{y-\cos\beta}\,, \end{aligned} \tag{3.21}$$

because, by (3.19), we know that $y \neq \cos\beta$.

Next, since $\cos^2\alpha + \sin^2\alpha = 1$, from (3.21) we have

$$x^2\cos^2\beta + y^2\sin^2\beta = (y-\cos\beta)^2. \tag{3.22}$$

Hence, simplifying the expression above, we find

$$\Big[(x^2-y^2-1)\cos\beta + 2y\Big]\cos\beta \,=0. \tag{3.23}$$

Since $\cos\beta \neq 0$ and (by (3.19)) $y \neq 0$, we deduce that:

$$x^2 - y^2 - 1 \neq 0, \tag{3.24}$$

and then

$$\cos\beta = \frac{-2y}{x^2-y^2-1}. \tag{3.25}$$

Noting that $x \neq 0, \cos\alpha$ (see (3.18)) by similar arguments we can derive that

$$y^2 - x^2 - 1 \neq 0 \tag{3.26}$$

and

$$\cos\alpha = \frac{-2x}{y^2-x^2-1}\,. \tag{3.27}$$

Finally, since (3.16) is equivalent to $\cos\alpha, \cos\beta \neq \pm 1$, from the expressions (3.25), (3.27) we deduce the conditions:

$$-1 < \frac{-2y}{x^2-y^2-1} < 1, \tag{3.28}$$

$$-1 < \frac{-2x}{y^2-x^2-1} < 1. \tag{3.29}$$

Summing up, if (3.10), (3.16) are verified and if $P_3 = {}^t(x,y,0)$ is given by formula (3.11), then $x, y \neq 0$ and the necessary conditions (3.24), (3.26) and (3.28), (3.29) are satisfied.

Remark 3.3 Note that $\cos\alpha, \cos\beta$ are uniquely determined and they are rational functions of x, y. We haven't similar expressions for $\sin\alpha, \sin\beta$. But we can easily see that

$$\frac{\sin\beta}{\sin\alpha} = -\,\frac{y^2-x^2-1}{x^2-y^2-1}\,. \tag{3.30}$$

Indeed, from (3.19) we have

$$\frac{\sin\beta}{\sin\alpha} = 1 - \frac{\cos\beta}{y}\,. \tag{3.31}$$

Then, by substituting (3.25) into (3.31), we get (3.30).

We will use this fact in proving that for $x, y \neq 0$ the necessary conditions (3.24), (3.26) and (3.28), (3.29) are also sufficient for a point $P_3 = {}^t(x,y,0)$ to be collinear with R_3, R_3' given by (3.9) for suitable α, β satisfying (3.10), (3.16). □

In order to better describe the solution region of the inequalities (3.24), (3.26) and (3.28), (3.29) we prove the following:

Lemma 3.4 *The inequalities (3.24), (3.26) and (3.28), (3.29) are verified if and only if*

$$|x| + |y| < 1 \quad \textit{or} \quad \big||x| - |y|\big| > 1. \tag{3.32}$$

Proof. We must show that the solution region of the inequalities (3.24), (3.26) and (3.28), (3.29) is given by the conditions (3.32). First of all, we can see that:

$$-1 < \frac{-2y}{x^2 - y^2 - 1} < 1 \quad \Longleftrightarrow \quad (\mathrm{I}) \begin{cases} \dfrac{x^2 - (y-1)^2}{x^2 - y^2 - 1} > 0 \\[2ex] \dfrac{x^2 - (y+1)^2}{x^2 - y^2 - 1} > 0 \end{cases} \tag{3.33}$$

and

$$-1 < \frac{-2x}{y^2 - x^2 - 1} < 1 \quad \Longleftrightarrow \quad (\mathrm{II}) \begin{cases} \dfrac{y^2 - (x-1)^2}{y^2 - x^2 - 1} > 0 \\[2ex] \dfrac{y^2 - (x+1)^2}{y^2 - x^2 - 1} > 0 \end{cases} \tag{3.34}$$

Both systems of inequalities (I) of (3.33) and (II) of (3.34) are invariant under symmetry with respect to the coordinate axes, i.e., on replacing (x, y) with $(\pm x, \pm y)$. Besides, it is evident that we can obtain system (II) of (3.34) from (I) of (3.33) by permutation of the variables x, y and vice versa. So it is sufficient to solve only one of them and for (x, y) in the first quadrant. For instance, let us consider system (II) of (3.34). For $x, y \geq 0$ we easily have:

$$(\mathrm{II}) \begin{cases} \dfrac{y^2 - (x-1)^2}{y^2 - x^2 - 1} > 0 \\[2ex] \dfrac{y^2 - (x+1)^2}{y^2 - x^2 - 1} > 0 \end{cases} \quad \Longleftrightarrow \quad (\mathrm{III}) \begin{cases} \dfrac{y - |x-1|}{y^2 - x^2 - 1} > 0 \\[2ex] \dfrac{y - x - 1}{y^2 - x^2 - 1} > 0 \end{cases} \tag{3.35}$$

because in the first quadrant

$$y^2 - (x-1)^2 > 0 \quad \Leftrightarrow \quad y - |x-1| > 0, \tag{3.36}$$

$$y^2 - (x+1)^2 > 0 \quad \Leftrightarrow \quad y - x - 1 > 0. \tag{3.37}$$

Thus it is enough to solve system (III) of (3.35) for $x, y \geq 0$. Noting that $y - |x-1| \geq y - x - 1$ if $x \geq 0$, we can see that for $x, y \geq 0$ the following hold:

(i) (x, y) is a solution of (III) such that $y^2 - x^2 - 1 > 0$ iff $y > x + 1$, that is $y - x > 1$;

(ii) (x, y) is a solution of (III) such that $y^2 - x^2 - 1 < 0$ iff $y < |x - 1|$, that is

$$x + y < 1 \quad \text{or} \quad y - x < -1. \tag{3.38}$$

Hence we deduce that the solution region, say Ω, of system (III) of (3.35) (that is, system (II) of (3.34)) in the first quadrant is given by the pairs $x, y \geq 0$ such that

$$x + y < 1 \quad \text{or} \quad |y - x| > 1. \tag{3.39}$$

Since Ω is symmetric with respect to x and y, it follows that Ω is also the solution region of system (I) of (3.33) in the first quadrant. Finally, taking into account the symmetry of both systems (I) of (3.33) and (II) of (3.34) with respect to the coordinate axes, the solution region of the inequalities (3.24), (3.26) and (3.28), (3.29) is given by the conditions (3.32). □

So far, we have proved that:

Lemma 3.5 *If (3.10), (3.16) hold and if* $P = {}^t(x, y, 0)$ *is given by (3.11), then* $x, y \neq 0$ *and*

$$|x| + |y| < 1 \quad \textit{or} \quad \big||x| - |y|\big| > 1. \tag{3.40}$$

The converse is also true:

Lemma 3.6 *If a point* $P = {}^t(x, y, 0)$ *has coordinates* $x, y \neq 0$ *such that (3.40) holds, then* P *is given by formula (3.11) with* α, β *satisfying (3.10), (3.16).*

Proof. By condition (3.40) and Lemma 3.4 we can set

$$\cos\alpha = \frac{-2x}{y^2 - x^2 - 1} \quad \text{and} \quad \cos\beta = \frac{-2y}{x^2 - y^2 - 1}. \tag{3.41}$$

We have $-1 <\ \cos\alpha, \cos\beta\ < -1$, i.e., $\sin\alpha, \sin\beta \neq 0$. Besides $\cos\alpha, \cos\beta \neq 0$ because we are assuming $x, y \neq 0$. Then the first two equations of (3.11) are satisfied with

$$t = \frac{y^2 - x^2 + 1}{2}. \tag{3.42}$$

With t as in (3.42) the third equation of (3.11) is verified iff

$$\frac{\sin\beta}{\sin\alpha} = -\frac{y^2 - x^2 - 1}{x^2 - y^2 - 1}. \tag{3.43}$$

But using (3.41) to write explicitly $\sin^2\alpha$ and $\sin^2\beta$, it follows that

$$\sin^2\alpha = \frac{g(x,y)}{(y^2 - x^2 - 1)^2} \quad \text{and} \quad \sin^2\beta = \frac{g(x,y)}{(x^2 - y^2 - 1)^2}, \tag{3.44}$$

where

$$g(x,y) \stackrel{\text{def}}{=} (x + y + 1)(x + y - 1)(x - y - 1)(x - y + 1) > 0, \tag{3.45}$$

if (3.40) holds. Thus (3.43) is verified iff

$$(\sin\alpha, \sin\beta) = \pm\left(\frac{\sqrt{g(x,y)}}{y^2 - x^2 - 1}, \frac{-\sqrt{g(x,y)}}{x^2 - y^2 - 1}\right). \tag{3.46}$$

Finally, it is immediate that (3.43) implies $\sin\alpha \neq \sin\beta$. So both conditions (3.10) and (3.16) are satisfied. □

Remark 3.7 It is easy to verify that (3.40) holds if and only if

$$(x+y+1)(x+y-1)(x-y-1)(x-y+1) > 0. \tag{3.47}$$

Thus the conditions (1.7) and (1.8) are equivalent. $\square$

As we have already observed,

$$x, y \neq 0 \ \text{ and } \ (3.10), (3.11) \quad \Longrightarrow \quad (3.16),$$

so we may conclude the following:

Let us suppose $x, y \neq 0$. Then system (3.11) with the conditions (3.10) is solvable $\Leftrightarrow$ (3.40) holds. Moreover, if $x, y \neq 0$ and (3.40) is verified, then

$$R_3 = \frac{1}{y^2 - x^2 - 1} \begin{pmatrix} -2x \\ 0 \\ \pm\sqrt{g(x,y)} \end{pmatrix}, \tag{3.48}$$

where $g(x,y)$ is the function defined by (3.45).

In particular, choosing the plus and then the minus sign in (3.48), we obtain two secondary Pohlke's projections Π, $\bar{\Pi}$ which are symmetric with respect to the plane ω. Noting that $x, y \neq 0 \Leftrightarrow OP_3 \nparallel OP_1, OP_2$, we have:

Lemma 3.8 *If (3.1) is verified and if $OP_3 \nparallel OP_1, OP_2$, then there exists a secondary Pohlke's projection for OP_1, OP_2, OP_3 if and only if*

$$\overrightarrow{OP_3} = x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}\,, \tag{3.49}$$

with x, y such that (3.40) holds. Besides, the secondary Pohlke's projection (when it exists) is unique up to symmetry with respect to the plane ω.

4 Proof of Theorem 1.4

(2) $\Rightarrow$ (1). Taking into account (2) of Lemma 2.6 and Lemma 2.9, we can say that:

if $\mathcal{E}$ is a secondary Pohlke's ellipse for OP_1, OP_2, OP_3, then $\Pi_{\mathcal{E}}$ is a secondary Pohlke's projection for OP_1, OP_2, OP_3.

Indeed, by (2) of Lemma 2.6, $\Pi_{\mathcal{E}}$ verifies the assumptions of (1) of Lemma 2.6. But, having $\mathcal{E} = \Pi_{\mathcal{E}}(S_{\mathcal{E}} \cap \pi_{\mathcal{E}})$ with $\mathcal{E} \neq \mathcal{E}_{\mathsf{P}}$, $\Pi_{\mathcal{E}}$ cannot be a Pohlke's projection for OP_1, OP_2, OP_3. Thus, by Lemma 2.9, $\Pi_{\mathcal{E}}$ must be a secondary Pohlke's projection for OP_1, OP_2, OP_3.

(1) $\Rightarrow$ (2). It follows from (1) of Lemma 2.6 and from Lemma 2.9. Indeed, let Π be a secondary Pohlke's projection for OP_1, OP_2, OP_3. By Definition 1.3 there are equal segments OR_1, OR_2, OR_3 such that (1.4), (1.5) and (1.6) hold. Then Π satisfies the conditions (i), (ii) and (iv) of (1) of Lemma 2.6 with $Q_i = R_i$ $(1 \leq i \leq 3)$ and, by Lemma 2.9, Π is not a Pohlke's projection for OP_1, OP_2, OP_3. Now, let us consider the ellipse

$$\mathcal{E} = \Pi(S \cap \pi), \tag{4.1}$$

where S is the sphere, centered at O, containing the points R_1, R_2, R_3 and π is the plane through O and perpendicular to the direction of Π. By (1) of Lemma 2.6, $\mathcal{E}$ circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Moreover, $\mathcal{E} \neq \mathcal{E}_{\mathsf{P}}$ because Π is not a Pohlke's projection. Thus, by Definition 1.3, $\mathcal{E}$ is a secondary Pohlke's ellipse for OP_1, OP_2, OP_3.

(1), (2) $\Leftrightarrow$ (3). To prove the equivalence of the conditions (1), (2) with (3) when OP_1, OP_2 and OP_3 are non-parallel, we resort to an appropriate circular case.

More precisely, let $N_1, N_2 \in \omega$ such that

$$ON_1 \perp ON_2 \quad \text{and} \quad |ON_1| = |ON_2| = 1. \tag{4.2}$$

Since $OP_1 \nparallel OP_2$, we may consider the affine transformation $\Phi : \omega \to \omega$ such that

$$\Phi(O + x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}\,) \stackrel{\text{def}}{=} O + x\,\overrightarrow{ON_1} + y\,\overrightarrow{ON_2} \quad \text{for} \quad x, y \in \mathbb{R}. \tag{4.3}$$

It is clear that $\Phi(P_1) = N_1$, $\Phi(P_2) = N_2$. Besides, if $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$, then

$$N_3 \stackrel{\text{def}}{=} \Phi(P_3) = O + h\,\overrightarrow{ON_1} + k\,\overrightarrow{ON_2}. \tag{4.4}$$

Hence

$$\overrightarrow{ON_3} = h\,\overrightarrow{ON_1} + k\,\overrightarrow{ON_2} \quad \text{and} \quad ON_3 \nparallel ON_1, ON_2, \tag{4.5}$$

because $OP_3 \nparallel OP_1, OP_2$ (i.e., $h, k \neq 0$).

As we already remarked after Definition 2.4, an affine transformation maps pairs of conjugate semi-diameters of a central conic into pairs of conjugate semi-diameters of the transformed conic. This means that $\Phi(\mathcal{E}_{P_1,P_2}) = \mathcal{E}_{N_1,N_2}$, $\Phi(\mathcal{E}_{P_2,P_3}) = \mathcal{E}_{N_2,N_3}$ and $\Phi(\mathcal{E}_{P_3,P_1}) = \mathcal{E}_{N_3,N_1}$. Further, if $\mathcal{E}$ is an ellipse centered at O which circumscribes the three ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$, then $\Phi(\mathcal{E})$ is an ellipse centered at O which circumscribes $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$ and $\mathcal{E}_{N_3,N_1}$.

Now let us suppose that (2) holds, namely that there exists a secondary Pohlke's ellipse $\mathcal{E}$ for OP_1, OP_2, OP_3. Then $\mathcal{E}$ and $\mathcal{E}_{\mathsf{P}}$ are two different ellipses centered at O which circumscribe $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Consequently

$$\Phi(\mathcal{E}) \quad \text{and} \quad \Phi(\mathcal{E}_{\mathsf{P}}) \tag{4.6}$$

are two different ellipses centered at O which circumscribes $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$, $\mathcal{E}_{N_3,N_1}$. Since the Pohlke's ellipse for ON_1, ON_2, ON_3 is unique, one of them must be a secondary Pohlke's ellipse for ON_1, ON_2, ON_3. Hence, having already proved that (1) $\Leftrightarrow$ (2), there exists a secondary Pohlke's projection for ON_1, ON_2, ON_3. By (4.2) and (4.5) we can apply Lemma 3.8 to ON_1, ON_2, ON_3. Thus we conclude that h, k must satisfy the condition (1.7).

Conversely, let us suppose that (3) (i.e., condition (1.7)) holds. Then, by Lemma 3.8, there exists a secondary Pohlke's projection for ON_1, ON_2, ON_2. By the equivalence (1) $\Leftrightarrow$ (2), we deduce the existence of a secondary Pohlke's ellipse for ON_1, ON_2, ON_3. Then we have two different ellipses centered at O, say $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_{\mathsf{P}}$ (with $\tilde{\mathcal{E}}_{\mathsf{P}}$ the Pohlke's ellipses for ON_1, ON_2, ON_2) which circumscribe $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$, $\mathcal{E}_{N_3,N_1}$. This means that

$$\Phi^{-1}(\tilde{\mathcal{E}}) \quad \text{and} \quad \Phi^{-1}(\tilde{\mathcal{E}}_{\mathsf{P}}) \tag{4.7}$$

are two different ellipses centered at O which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Since the Pohlke's ellipse is unique, one of them must be a secondary Pohlke's ellipse for OP_1, OP_2, OP_3. Thus, we have proved that (2) holds.

Uniqueness. Suppose OP_1, OP_2, OP_3 are non-parallel and condition (3) holds.

Having already proved that (1) $\Leftrightarrow$ (2), it is enough to demonstrate the uniqueness of the secondary Pohlke's ellipse. By contradiction, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two distinct secondary Pohlke's ellipses for OP_1, OP_2, OP_3, then

$$\Phi(\mathcal{E}_1),\ \Phi(\mathcal{E}_2) \quad \text{and} \quad \Phi(\mathcal{E}_{\mathrm{P}}) \tag{4.8}$$

are three distinct ellipses circumscribing $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$, $\mathcal{E}_{N_3,N_1}$. Since the Pohlke's ellipse for ON_1, ON_2, ON_3 is unique, two of them must be secondary Pohlke's ellipses for ON_1, ON_2, ON_3. But if $\hat{\mathcal{E}}$ and $\check{\mathcal{E}}$ are these two ellipses, then

$$\Pi_{\hat{\mathcal{E}}} \quad \text{and} \quad \Pi_{\check{\mathcal{E}}} \tag{4.9}$$

must be two distinct (that is, $\Pi_{\hat{\mathcal{E}}} \neq \Pi_{\check{\mathcal{E}}}, \bar{\Pi}_{\check{\mathcal{E}}}$; see (a) of Remark 2.3) secondary Pohlke's projections for the three segments ON_1, ON_2, ON_3 and, by Lemma 3.8, this contradicts the uniqueness property of the secondary Pohlke's projection in the circular case. □

5 Proof of Theorem 1.5

Let us suppose, for instance, that $OP_1 \nparallel OP_2$ and $OP_2 \parallel OP_3$. With the same notations of the proof of Theorem 1.4, we consider the affine transformation $\Phi : \omega \to \omega$ defined in (4.2), (4.3). Then we distinguish two cases:

1) $\overrightarrow{OP_3} = \pm\overrightarrow{OP_2}$. We have $\overrightarrow{ON_3} = \pm\overrightarrow{ON_2}$ and by Lemma 3.2 there are infinite, distinct secondary Pohlke's projections for ON_1, ON_2, ON_3. Hence, by the equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.4, there are infinite, distinct secondary Pohlke's ellipses $\tilde{\mathcal{E}}$ circumscribing $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$ and $\mathcal{E}_{N_3,N_1}$. Applying the affine transformation Φ^{-1}, we find infinite secondary Pohlke's ellipses $\mathcal{E} = \Phi^{-1}(\tilde{\mathcal{E}})$ for OP_1, OP_2, OP_3 and finally infinite, distinct secondary Pohlke's projections.

2) $\overrightarrow{OP_3} \neq \pm\overrightarrow{OP_2}$. We argue by contradiction: if there is a secondary Pohlke's projection for OP_1, OP_2, OP_3 then, by the equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.4, there is also a secondary Pohlke's ellipse $\mathcal{E}$ for OP_1, OP_2, OP_3. As in the proof of (1), (2) $\Leftrightarrow$ (3) of Theorem 1.4, applying the transformation Φ we deduce the existence of a secondary Pohlke's ellipse $\tilde{\mathcal{E}}$ for ON_1, ON_2, ON_3. Hence, once again by the equivalence (1) $\Leftrightarrow$ (2), we conclude that there exists a secondary Pohlke's projection for ON_1, ON_2, ON_3. But this fact contradicts Lemma 3.2, because $\overrightarrow{ON_3} \parallel \overrightarrow{ON_2}$ and $\overrightarrow{ON_3} \neq \pm\overrightarrow{ON_2}$. □

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