A proof of Pohlke's theorem with an analytic determination of the reference trihedron

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Abstract

By elementary arguments of linear algebra and vector algebra we give here a proof of Pohlke's fundamental theorem of oblique axonometry. We also give simple explicit formulae for the reference trihedrons (Pohlke matrices) and the corresponding directions of projection onto the drawing plane.

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1 Introduction

The famous Pohlke's fundamental theorem of oblique axonometry asserts that:

three arbitrary straight line segments OP_1, OP_2, OP_3 in a plane, originating from a point O and which are not contained in a line, can be considered as the parallel projection of three edges OQ_1, OQ_2, OQ_3 of a cube.

Or, with the words of H. Steinhaus ([11], p. 170),

one can draw any three (not all parallel) segments from one point, complete the figure with parallel segments, and consider it as a (generally oblique) projection of a cube.

K.W. Pohlke formulated this theorem in 1853 and published it in 1860, without demonstration, in the first part of his textbook on descriptive geometry [9]. The first elementary rigorous proof was given by H.A. Schwarz [10] in 1864, at that time a student of Pohlke.

Subsequently, as remarked by D.J. Struik ([12], p. 240), several proofs have been given, synthetic and analytic, none of which is simple because they also give the method by which one can construct the direction of projection. See among the others [3, 4, 7, 1, 6, 8, 2].

Here we prove Pohlke's theorem by reducing it to a particular case in which the direction of projection coincides with that of one of the edges of the cube. To achieve this simplification we restate Pohlke's theorem as a theorem of linear algebra and then we make a simple observation based on the fact that for a matrix *row*-rank and *column*-rank are equal. More precisely, if we introduce a cartesian system of coordinates such that

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad P_i = \begin{pmatrix} x_i \\ y_i \\ 0 \end{pmatrix} \quad (1 \le i \le 3) \tag{1.1}$$

are points of the plane $\{z = 0\}$ then Pohlke's theorem can be reformulated as follows:

Theorem 1.1 *If the matrix*

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} = (A^1, A^2, A^3) \tag{1.2}$$

has rank 2*, then there exist a matrix* B *with orthogonal columns of equal norm,*

$$B = \begin{pmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \\ z'_1 & z'_2 & z'_3 \end{pmatrix} = (B^1, B^2, B^3)\,, \tag{1.3}$$

and a parallel projection Π *onto the plane* $\{z = 0\}$ *such that* $\Pi(B^i) = A^i \quad (1 \le i \le 3\,)$.

Now, see Lemma 2.1, the columns of A can be obtained by a parallel projection of the columns of the matrix B if and only if the rows of A can be obtained by a parallel projection of the rows of the matrix B. But, since the third row of A has zero entries, the direction of this last projection is the same of the third row of B.

As a by-product, we finally obtain explicit formulae for the reference trihedron and the direction of projection. For instance, for B, Π (which are *not* unique) we may take:

$$B = \frac{1}{1+\alpha^2+\beta^2} \begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3-y_2x_3}{\varrho} & \frac{y_1x_3-x_1y_3}{\varrho} & \frac{x_1y_2-y_1x_2}{\varrho} \end{pmatrix} \tag{1.4}$$

and

$$\Pi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+\alpha z \\ y+\beta z \\ 0 \end{pmatrix}, \tag{1.5}$$

with α, β, ϱ defined by (3.6), (3.10), (3.21), (3.22) below from the rows A_1, A_2 of A. See also (4.6), (4.8) and the simple examples at the end of Chapter 4.

The fact that the proof becomes easier if one has some information about the direction of projection is particularly evident when Π is, *a priori*, the orthogonal projection onto the plane $\{z = 0\}$. More precisely, denoting with A_1, A_2, A_3 the rows of the matrix A given by (1.2), we can reformulate the Gauss' theorem of orthogonal axonometry as follows:

Proposition 1.2 A_1, A_2 *are nonzero orthogonal rows of equal norm, that is*

$$\|A_1\| = \|A_2\| \neq 0 \quad \textit{with} \quad A_1 \perp A_2\,, \tag{1.6}$$

if and only if there exists a matrix B *with nonzero orthogonal columns of equal norm such that*

$$\Pi_\perp(B^i) = A^i \quad (1 \le i \le 3) \tag{1.7}$$

where $\Pi_\perp : \mathbf{R}^3 \to \{z = 0\}$ *is the orthogonal projection.*

Proof: In fact, assuming (1.6) and setting $\rho = \|A_1\| = \|A_2\|$, we may define the row vector $B_3 = (z'_1, z'_2, z'_3)$ as

$$B_3 = \frac{1}{\rho}\, A_1 \wedge A_2 \quad \text{or} \quad B_3 = -\frac{1}{\rho}\, A_1 \wedge A_2\,. \tag{1.8}$$

Namely, we choose B_3 equal to (z_1, z_2, z_3) or $(-z_1, -z_2, -z_3)$ where

$$z_1 = \frac{1}{\rho}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \qquad z_2 = -\frac{1}{\rho}\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \qquad z_3 = \frac{1}{\rho}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}. \tag{1.9}$$

Then, $B_3 \perp A_1, A_2$ and $\|B_3\| = \rho$. Hence, setting $B_1 = A_1$, $B_2 = A_2$, the matrix

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z'_1 & z'_2 & z'_3 \end{pmatrix} \tag{1.10}$$

has orthogonal rows of norm ρ. This means that B is a multiple of an orthogonal matrix and thus it follows that the columns B^i $(1 \le i \le 3)$ are orthogonal and of norm ρ. Finally, if $\Pi_{\perp} : \mathbf{R}^3 \to \{z = 0\}$ is the orthogonal projection onto the plane $\{z = 0\}$, we clearly have (1.7).

Conversely, if there exists a matrix $B = (B^1, B^2, B^3)$ such that (1.7) holds, than $B_1 = A_1$ and $B_2 = A_2$. If, in addition, the columns B^1, B^2, B^3 are nonzero, orthogonal and of equal norm then B is a multiple of an orthogonal matrix. Thus, the rows A_1 and A_2 are nonzero, orthogonal and of equal norm. □

Remark 1.3 In the proof of Prop. 1.2 the row B_3 is necessarily given by one of the expressions of (1.8), since (1.7) implies that $B_1 = A_1$, $B_2 = A_2$. Hence, there are exactly two distinct possibilities for B: one with $B_3 = \frac{1}{\rho}\, A_1 \wedge A_2$ and the other with $B_3 = -\frac{1}{\rho}\, A_1 \wedge A_2$. □

We may say that in case of orthogonal projection "Pohlke's problem" (namely, that of finding a matrix B and a projection Π with the required properties) has exactly two distinct solutions.

More generally, to consider the question of the multiplicity of solutions of "Pohlke's problem", we give the following definition:

Definition 1.4 (Pohlke matrix) *Let A be a matrix of rank 2 as in* (1.2). *We say that a matrix B is a "Pohlke matrix" for A if B has orthogonal columns of equal norm and there exists a parallel projection $\Pi : \mathbf{R}^3 \to \{z = 0\}$ such that $\Pi(B^i) = A^i$, $1 \le i \le 3$.*

Besides, we say that two Pohlke matrices B, $\tilde{B}$ are "conjugate" if they correspond to the same parallel projection $\Pi : \mathbf{R}^3 \to \{z = 0\}$, that is $\Pi(B^i) = \Pi(\tilde{B}^i) = A^i$, $1 \le i \le 3$.

Then, we have:

Corollary 1.5 *Under the assumption of Theorem 1.1, in case of oblique projection* (*i.e., non-orthogonal projection*) *there are exactly two couples of conjugate Pohlke matrices that correspond to two distinct, oblique, directions of projection.*

See the explicit formulae (4.6) and (4.8) below.

2 Some facts of linear algebra

We give here two simple results of linear algebra. The first is the key lemma that we need in the proof of Pohlke's theorem. The second one is a standard fact on orthogonal transition matrices; see [5]. We need it to compute the direction of the projection $\Pi : \mathbf{R}^3 \to \{z = 0\}$ and the mutually orthogonal column vectors B^i such that $\Pi(B^i) = A^i$.

Let A be the real 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \doteqdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \doteqdot (A^1, A^2, A^3) \tag{2.1}$$

where A_i, A^i are, respectively, row and column vectors of A:

$$A_i = (a_{i1},\, a_{i2},\, a_{i3})\,, \quad A^i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \end{pmatrix} \quad (1 \le i \le 3)\,. \tag{2.2}$$

We indicate with $\mathbb{V}_A$ and $\mathbb{V}^A$ the row and column spaces of A, namely

$$\mathbb{V}_A = \mathrm{span}\big\{A_1, A_2, A_3\big\}\,, \quad \mathbb{V}^A = \mathrm{span}\big\{A^1, A^2, A^3\big\}\,. \tag{2.3}$$

If $\mathrm{rank}(A) = 2$, then $\mathbb{V}^A$ and $\mathbb{V}_A$ are 2-dimensional subspaces (i.e., planes throw the origin) of $\mathbf{R}^3$. Given a column vector U, $U \nparallel \mathbb{V}^A$, we may consider the parallel projection $\Pi^U : \mathbf{R}^3 \to \mathbb{V}^A$, in the direction of U, by setting

$$\Pi^U(V) = \mathbb{V}^A \cap \{V + t\,U : t \in \mathbf{R}\} \tag{2.4}$$

for any column vector $V \in \mathbf{R}^3$. Since $\mathbb{V}^A$ is a plane throw the origin, Π^U is a *linear* map.

In the same way we can define the parallel projection $\Pi_W : \mathbf{R}^3 \to \mathbb{V}_A$ in the direction of a given row vector W, if $W \nparallel \mathbb{V}_A$.

When the direction of projection is not specified, we write Π^* (Π_*) for column (row) projections onto $\mathbb{V}^A$ ($\mathbb{V}_A$).

Lemma 2.1 *Let A and B be 3×3 matrices such that* $\mathrm{rank}(A) = 2$ *and* $\mathrm{rank}(B) = 3$.

Then the rows of A can be obtained by a parallel projection $\Pi_* : \mathbf{R}^3 \to \mathbb{V}_A$ of the rows of the matrix B if and only if the columns of A can be obtained by a parallel projection $\Pi^* : \mathbf{R}^3 \to \mathbb{V}^A$ of the columns of the matrix B.

Proof: Let Π_* be a parallel projection onto $\mathbb{V}_A$ such that

$$\Pi_*(B_i) = A_i \quad \text{for} \quad 1 \le i \le 3\,. \tag{2.5}$$

Then we have

$$\text{row-rank}\,(B - A) = \text{row-rank} \begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \\ B_3 - A_3 \end{pmatrix} = 1, \tag{2.6}$$

because the row vectors $B_i - A_i$ are parallel to the direction of the projection Π_*. Since row-rank and column-rank of a matrix are equal ([5], p. 81) this implies that

$$\text{column-rank}\,(B - A) = \text{column-rank}\,(B^1 - A^1,\, B^2 - A^2,\, B^3 - A^3) = 1. \tag{2.7}$$

Thus there exists a column vector $U \in \mathbf{R}^3$ such that

$$(B^i - A^i) \parallel U \quad \text{for} \quad 1 \le i \le 3\,. \tag{2.8}$$

Moreover, observe that

$$U \notin \mathbb{V}^A \quad (\text{that is } U \nparallel \mathbb{V}^A) \tag{2.9}$$

because $\operatorname{rank}(B) = 3$, while $\operatorname{rank}(A) = 2$.

Hence, for all column vector $V \in \mathbf{R}^3$ the line $\{V + Ut : t \in \mathbf{R}\}$ has one and only one intersection with the plane $\mathbb{V}^A$. This permits us to define the projection $\Pi^* : \mathbf{R}^3 \to \mathbb{V}^A$ in the direction of the column vector U by setting

$$\Pi^*(V) = \mathbb{V}^A \cap \{V + Ut : t \in \mathbf{R}\} \quad \text{for} \quad V \in \mathbf{R}^3. \tag{2.10}$$

Since we clearly have $\Pi^*(B^i) = A^i$ $(1 \le i \le 3)$, this proves the first part of the lemma. The converse can be proved similarly. □

Remark 2.2 The common value of row-rank and column-rank is obvious in the case of the previous proof. In fact, it is evident that if $B_i - A_i = \lambda_i W$ for $1 \le i \le 3$ (W a given row vector, $\lambda_i \in \mathbf{R}$) then $(B^i - A^i) \parallel \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$ $(1 \le i \le 3)$. Thus we can choose $U = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$. □

Having proved Lemma 2.1, and taking into account that if B is a square matrix then

B has nonzero orthogonal columns of equal norm $\Leftrightarrow$ B is a nonzero multiple of an orthogonal matrix $\Leftrightarrow$ B has nonzero orthogonal rows of equal norm,

we can immediately restate Definition 1.4 in the following equivalent form:

Definition 2.3 *Let A be a 3×3 matrix of rank 2. We say that a 3×3 matrix B is a "Pohlke matrix" for A if B is a multiple of an orthogonal matrix and there exist parallel projections $\Pi_* : \mathbf{R}^3 \to \mathbb{V}_A$ and $\Pi^* : \mathbf{R} \to \mathbb{V}^A$ such that $\Pi_*(B_i) = A_i$ and $\Pi^*(B^i) = A^i$ for $1 \le i \le 3$.*

Next, we recall a standard result concerning the existence of orthogonal transition matrices for two given bases of $\mathbf{R}^3$. We state Lemma 2.5 below in terms of row vectors but, by transposition, a similar result holds for column vectors.

Definition 2.4 *We denote by E_i, $1 \le i \le 3$, the standard base of row vectors:*

$$E_1 = (1,0,0)\,, \quad E_2 = (0,1,0)\,, \quad E_3 = (0,0,1)\,. \tag{2.11}$$

Lemma 2.5 *Let $\{A'_1, A'_2, A'_3\}$ and $\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3\}$ be two sets of linearly independent row vectors (i.e., two bases of $\mathbf{R}^3$) such that*

$$A'_i \cdot A'_j = \widetilde{A}_i \cdot \widetilde{A}_j \qquad (1 \le i,\, j \le 3)\,. \tag{2.12}$$

Then, there exists a unique orthogonal transition matrix $\mathcal{T}$ such that $A'_i = \widetilde{A}_i\,\mathcal{T}$ $(1 \le i \le 3)$.

Proof: Uniqueness is quite obvious. To determine $\mathcal{T}$, let us define the nonsingular matrices

$$\mathcal{G} = \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix}, \quad \widetilde{\mathcal{G}} = \begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \\ \widetilde{A}_3 \end{pmatrix}. \tag{2.13}$$

We have

$$E_i\,\mathcal{G} = A_i', \quad E_i\,\widetilde{\mathcal{G}} = \widetilde{A}_i \quad 1 \le i \le 3. \tag{2.14}$$

Then, setting $\mathcal{T} = \widetilde{\mathcal{G}}^{-1}\,\mathcal{G}$, it is clear that

$$\widetilde{A}_i\,\mathcal{T} = \widetilde{A}_i\,\widetilde{\mathcal{G}}^{-1}\,\mathcal{G} = E_i\,\mathcal{G} = A_i'. \tag{2.15}$$

On the other hand, using (2.12) and the linearity of the map $X \longmapsto X\,\mathcal{T}$, it is easy to see that for all row vectors $X \in \mathbf{R}^3$ one has $\|X\,\mathcal{T}\| = \|X\|$.

In fact, since $\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3\}$ is a base of $\mathbf{R^3}$, there exist unique scalars λ_1, λ_2, $\lambda_3 \in \mathbf{R}$ such that $X = \lambda_1\widetilde{A}_1 + \lambda_2\widetilde{A}_2 + \lambda_3\widetilde{A}_3$. Then, by (2.12), we have

$$\begin{aligned} \|X\mathcal{T}\|^2 = \|\sum \lambda_i A_i'\|^2 = \sum\lambda_i\lambda_j(A_i' \cdot A_j') \\ = \sum \lambda_i\lambda_j(\widetilde{A}_i \cdot \widetilde{A}_j) = \|\sum \lambda_i\widetilde{A}_i\|^2 = \|X\|^2, \end{aligned} \tag{2.16}$$

and it is well known that this is equivalent to the orthogonality of the matrix $\mathcal{T}$. See [5]. □

3 Proof of Pohlke's theorem

We prove here Pohlke's theorem as stated in Theorem 1.1. With the notations of the previous sections, in the following we suppose

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with} \quad \operatorname{rank}(A) = 2, \tag{3.1}$$

i.e., the rows A_1 and A_2 are linearly independent and $\mathbb{V}^A$ is the plane $\{z = 0\}$. By Lemma 2.1, it is sufficient to find a matrix

$$B = \begin{pmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \\ z_1' & z_2' & z_3' \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{3.2}$$

with orthogonal rows B_1, B_2, B_3 of equal norm, and a parallel projection $\Pi_W : \mathbf{R}^3 \to \mathbb{V}_A$, with $W \notin \mathbb{V}_A$, such that $\Pi_W(B_i) = A_i$ for $1 \le i \le 3$. This means that

$$A_i = B_i + \lambda_i W \quad (1 \le i \le 3) \tag{3.3}$$

where λ_i are suitable real coefficients. Since $A_3 = (0, 0, 0)$, it is clear that the projection Π_W must be parallel to B_3. Hence, we may assume $W = B_3$. Then, it is enough to prove that there exist B_1, B_2, B_3 orthogonal and of equal norm such that

$$\begin{cases} A_1 = B_1 + \lambda_1 B_3 \\ A_2 = B_2 + \lambda_2 B_3 \end{cases} \tag{3.4}$$

for suitable scalars $\lambda_1, \lambda_2 \in \mathbf{R}$. To this aim we first consider the following:

Auxiliary problem: *Let $\{E_1, E_2, E_3\}$ be the standard base of row vectors of Definition 2.4, and let $\gamma \in (0,\pi)$, $\lambda > 0$ be given parameters. We look for α, $\beta \in \mathbf{R}$ such that:*

$$\begin{cases} \dfrac{\|E_1 + \alpha E_3\|}{\|E_2 + \beta E_3\|} = \lambda \,, \\ \\ \dfrac{(E_1 + \alpha E_3)\cdot(E_2 + \beta E_3)}{\|E_1 + \alpha E_3\|\,\|E_2 + \beta E_3\|} = \cos\gamma \,. \end{cases} \tag{3.5}$$

Before proving the solvability of (3.5) it is worthwhile to define the quantity:

Definition 3.1 *For $(\gamma, \lambda) \in (0,\pi) \times (0,+\infty)$ we set*

$$\eta = \eta(\lambda,\gamma) \doteqdot \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma} \,. \tag{3.6}$$

Then, for $(\gamma, \lambda) \in (0,\pi) \times (0,+\infty)$, we have:

$$\eta(\lambda,\gamma) \geq \eta\left(\lambda, \frac{\pi}{2}\right) = \frac{\lambda^2 + 1 + |\lambda^2 - 1|}{2\lambda^2} = \begin{cases} 1/\lambda^2 & \text{if} \quad 0 < \lambda \leq 1 \\ 1 \quad \text{if} & \lambda \geq 1 \end{cases} \tag{3.7}$$

with strict inequality if $\gamma \neq \frac{\pi}{2}$. In particular, $\eta(\gamma,\lambda)$ satisfies the following:

$$\begin{aligned} &(i) \qquad \eta \geq 1\,, \ \ \eta\lambda^2 \geq 1 \quad \text{for all} \quad (\gamma,\lambda) \in (0,\pi)\times(0,+\infty)\,, \\ &(ii) \qquad \eta = 1 \quad \Leftrightarrow \quad (\gamma,\lambda) \in \left\{\frac{\pi}{2}\right\} \times [1,+\infty)\,, \\ &(iii) \quad \eta\lambda^2 = 1 \quad \Leftrightarrow \quad (\gamma,\lambda) \in \left\{\frac{\pi}{2}\right\} \times (0,1]\,. \end{aligned} \tag{3.8}$$

Finally, for simplicity of writing, we also introduce a "signum" function:

$$\operatorname{sgn}(t) \doteqdot \begin{cases} 1 & \text{if} \quad t \geq 0 \\ -1 & \text{if} \quad t < 0 \end{cases} \tag{3.9}$$

Lemma 3.2 *Assume that $(\gamma, \lambda) \in (0,\pi) \times (0,+\infty)$. Then the real solutions of (3.5) are*

$$(\alpha,\beta) = \pm\left(\sqrt{\eta\,\lambda^2 - 1}\ ,\ \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\ \right). \tag{3.10}$$

Thus, for $(\gamma,\lambda) \neq (\frac{\pi}{2}, 1)$ system (3.5) has two real, distinct solutions; for $(\gamma,\lambda) = (\frac{\pi}{2}, 1)$ the only solution is $(\alpha,\beta) = (0,0)$.

Proof: It is clear that system (3.5) is equivalent to the following

$$\begin{cases} 1 + \alpha^2 = \lambda^2(1+\beta^2) \\ \\ \alpha\beta = \sqrt{1+\alpha^2}\,\sqrt{1+\beta^2}\ \cos\gamma \,. \end{cases} \tag{3.11}$$

Multiplying the first equation of (3.11) by $(1+\beta^2)$ one easily sees that

$$\begin{aligned} \alpha^2\beta^2 &= \lambda^2(1+\beta^2)^2 - 1 - \alpha^2 - \beta^2 \\ &= \lambda^2(1+\beta^2)^2 - (\lambda^2+1)(1+\beta^2) + 1\,. \end{aligned} \tag{3.12}$$

On the other hand, squaring the other equation of (3.11), we have

$$\begin{aligned} \alpha^2\beta^2 &= (1+\alpha^2)\,(1+\beta^2)\,\cos^2\gamma \\ &= \lambda^2\,(1+\beta^2)^2\,\cos^2\gamma\,. \end{aligned} \tag{3.13}$$

Hence, from (3.12) and (3.13), we deduce that $1+\beta^2$ satisfies the second order equation

$$\sin^2\gamma\,\lambda^2(1+\beta^2)^2 - (\lambda^2+1)(1+\beta^2) + 1 = 0\,. \tag{3.14}$$

Solving (3.14), we obtain

$$1+\beta^2 = \frac{\lambda^2+1 \pm \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}}{2\lambda^2\sin^2\gamma}\,. \tag{3.15}$$

But, for all $\gamma \in (0,\pi)$, $\lambda > 0$, we have

$$\begin{aligned} \frac{\lambda^2+1 - \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}}{2\lambda^2\sin^2\gamma} &= \frac{2}{\lambda^2+1+\sqrt{(\lambda^2+1)^2-4\lambda^2\sin^2\gamma}} \\ &\le \frac{2}{\lambda^2+1+|\lambda^2-1|} \;\le\; 1\,, \end{aligned} \tag{3.16}$$

with equality only when $(\gamma,\lambda) \in \left\{\frac{\pi}{2}\right\} \times (0,1]$. Thus, taking into account $(i), (iii)$ of (3.8), for $(\gamma,\lambda) \notin \left\{\frac{\pi}{2}\right\} \times (0,1]$ we have

$$1+\beta^2 = \frac{\lambda^2+1+\sqrt{(\lambda^2+1)^2-4\lambda^2\sin^2\gamma}}{2\lambda^2\sin^2\gamma} = \eta \ge 1\,, \tag{3.17}$$

$$1+\alpha^2 = \frac{\lambda^2+1+\sqrt{(\lambda^2+1)^2-4\lambda^2\sin^2\gamma}}{2\sin^2\gamma} = \eta\lambda^2 > 1\,. \tag{3.18}$$

By the second equation of (3.11) $\alpha\beta > 0$ if $\gamma \in (0,\frac{\pi}{2})$, while $\alpha\beta < 0$ if $\gamma \in (\frac{\pi}{2},\pi)$; furthermore, by $(ii), (iii)$ of (3.8) and (3.17), (3.18), we have $\alpha^2 = \eta\lambda^2 - 1 > 0$, $\beta^2 = \eta - 1 = 0$ for $\gamma = \frac{\pi}{2}$, $\lambda > 1$. It follows that for $(\gamma,\lambda) \notin \{\frac{\pi}{2}\} \times (0,1]$ there are exactly two distinct solutions:

$$(\alpha,\beta) = \begin{cases} \pm\left(\sqrt{\eta\lambda^2-1},\,\sqrt{\eta-1}\,\right) & \text{if} \quad 0<\gamma<\frac{\pi}{2}\,, \quad \lambda>0 \\ \pm\left(\sqrt{\eta\lambda^2-1},\,\sqrt{\eta-1}\,\right) & \text{if} \quad \gamma=\frac{\pi}{2}\,, \quad \lambda>1 \\ \pm\left(\sqrt{\eta\lambda^2-1},\,-\sqrt{\eta-1}\,\right) & \text{if} \quad \frac{\pi}{2}<\gamma<\pi\,, \quad \lambda>0 \end{cases} \tag{3.19}$$

On the other hand, for $(\gamma,\lambda) \in \{\frac{\pi}{2}\} \times (0,1]$ it is immediate to verify that (3.11) is satisfied if and only if $\alpha = 0$ and $\beta^2 = \frac{1}{\lambda^2} - 1$. Thus, by (iii) of (3.8), we can write again

$$(\alpha,\beta) = \pm(\sqrt{\eta\lambda^2-1}\,, \sqrt{\eta-1}\,)\,, \tag{3.20}$$

and (by (*ii*) of (3.8)) we find two distinct solutions unless $(\gamma,\lambda)=(\frac{\pi}{2},1)$. It is now clear that (3.19), (3.20) can be summarized by formula (3.10) for all $(\gamma,\lambda)\in(0,\pi)\times(0,+\infty)$.

To conclude, it remains to verify that (3.10) gives, effectively, solutions of both equations of (3.5). The first one, $1+\alpha^2=\lambda^2(1+\beta^2)$, is obviously satisfied by (3.10). The second one, $\alpha\beta=\sqrt{1+\alpha^2}\,\sqrt{1+\beta^2}\,\cos\gamma$, is certainly satisfied for $\gamma=\frac{\pi}{2}$, because $\alpha=0$ or $\beta=0$. For $\gamma\neq\frac{\pi}{2}$ the sign of "$\alpha\beta$" is the same of "$\cos\gamma$"; thus, it enough to show the $\alpha^2\beta^2=(1+\alpha^2)(1+\beta^2)\cos^2\gamma$, which is equivalent to $(1+\alpha^2)(1+\beta^2)\sin^2\gamma=(1+\alpha^2)+(1+\beta^2)-1$. This can be easily verified by substituting the expressions (3.17), (3.18) for $1+\beta^2$, $1+\alpha^2$. □

Now, to prove the solvability of (3.4), we apply Lemma 2.5 and Lemma 3.2. To begin with, we set the values of the parameters $\gamma\in(0,\pi)$, $\lambda>0$ of system (3.5):

$$\gamma \doteqdot \arccos\left(\frac{A_1\cdot A_2}{\|A_1\|\|A_2\|}\right),\qquad \lambda\doteqdot\frac{\|A_1\|}{\|A_2\|}\,. \tag{3.21}$$

Applying Lemma 3.2, we choose

$$(\alpha,\beta)$$

a real solution of system (3.5), and then we introduce the "scale" factor $\varrho\in(0,+\infty)$:

Definition 3.3

$$\varrho\doteqdot\frac{\|A_1\|}{\sqrt{1+\alpha^2}}=\frac{\|A_1\|}{\lambda\sqrt{\eta}}=\frac{\|A_2\|}{\sqrt{\eta}}=\frac{\|A_2\|}{\sqrt{1+\beta^2}}\,. \tag{3.22}$$

Next, we look for an isometry which transforms the couple of vectors $\varrho(E_1+\alpha E_3)$, $\varrho(E_2+\beta E_3)$ into the couple A_1, A_2. To this aim, it is clear that there are only two possibilities: we consider the sets of linearly independent row vectors $\{A_1',A_2',A_3'\}$ and $\{\widetilde{A}_1,\widetilde{A}_2,\widetilde{A}_3\}$ defined by

$$\begin{aligned} A_1'&=A_1 \\ A_2'&=A_2 \\ A_3'&=\pm\,\varrho^{-1}A_1\wedge A_2\end{aligned}\qquad\text{and}\qquad \begin{aligned}\widetilde{A}_1&=\varrho(E_1+\alpha E_3)=(\varrho,0,\varrho\alpha)\\ \widetilde{A}_2&=\varrho(E_2+\beta E_3)=(0,\varrho,\varrho\beta)\\ \widetilde{A}_3&=\varrho^{-1}\widetilde{A}_1\wedge\widetilde{A}_2=(-\varrho\alpha,-\varrho\beta,\varrho)\end{aligned} \tag{3.23}$$

where for A_3' we may take, indifferently, the sign "$+$" or "$-$". From (3.21), (3.22) we have

$$A_i'\cdot A_j'=\widetilde{A}_i\cdot\widetilde{A}_j\qquad(1\le i,\,j\le 3)\,. \tag{3.24}$$

Hence, by Lemma 2.5, there exists a unique orthogonal, transition matrix $\mathcal{T}$ such that

$$A_i'=\widetilde{A}_i\,\mathcal{T}\quad(1\le i\le 3)\,. \tag{3.25}$$

In particular, since $\mathcal{T}$ is orthogonal, setting

$$B_i\doteqdot\varrho\,E_i\,\mathcal{T}\quad(1\le i\le 3) \tag{3.26}$$

we have $\|B_i\|=\varrho$ and $B_i\perp B_j$ for $i\neq j$. Besides (3.23), (3.25) give

$$\begin{aligned}A_1&=A_1'=\varrho E_1\,\mathcal{T}+\alpha(\varrho E_3\,\mathcal{T})=B_1+\alpha B_3\\ A_2&=A_2'=\varrho E_2\,\mathcal{T}+\beta(\varrho E_3\,\mathcal{T})=B_2+\beta B_3\,.\end{aligned} \tag{3.27}$$

Thus, we have proved that (3.4) is solvable.

4 Determination of Pohlke matrices

It is clear that the Pohlke matrix B, with the rows B_1, B_2, B_3 defined in (3.26), coincides with $\varrho\mathcal{T}$ where $\mathcal{T}$ is the transition matrix given by Lemma 2.5. Namely, we have

$$B = \varrho\mathcal{T} = \varrho\,\widetilde{\mathcal{G}}^{-1}\mathcal{G}\,, \tag{4.1}$$

where $\mathcal{G}$, $\widetilde{\mathcal{G}}$ are 3×3 matrices defined as in (2.13). More precisely, we have:

$$\widetilde{\mathcal{G}} = \varrho \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ -\alpha & -\beta & 1 \end{pmatrix}, \tag{4.2}$$

$$\widetilde{\mathcal{G}}^{-1} = \frac{1}{\varrho(1+\alpha^2+\beta^2)} \begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix} \tag{4.3}$$

where (α,β) is any real solution of (3.5). For $\mathcal{G}$ there are two possibilities:

$$\mathcal{G} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3-y_2x_3}{\varrho} & \frac{y_1x_3-x_1y_3}{\varrho} & \frac{x_1y_2-y_1x_2}{\varrho} \end{pmatrix} \tag{4.4}$$

or

$$\mathcal{G} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3-y_2x_3}{-\varrho} & \frac{y_1x_3-x_1y_3}{-\varrho} & \frac{x_1y_2-y_1x_2}{-\varrho} \end{pmatrix}. \tag{4.5}$$

In conclusion, we finally obtain

$$B = \frac{1}{1+\alpha^2+\beta^2} \begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3-y_2x_3}{\pm\varrho} & \frac{y_1x_3-x_1y_3}{\pm\varrho} & \frac{x_1y_2-y_1x_2}{\pm\varrho} \end{pmatrix} \tag{4.6}$$

where (α,β) is any solution of (3.5).

Remark 4.1 Note that $\|B^1\| = \|B^2\| = \|B^3\| = \varrho$, because $\widetilde{\mathcal{G}}^{-1}\mathcal{G}$ is an orthogonal matrix. □

The direction of projection.

It is now easy to find the direction of the projection corresponding to the two Pohlke matrices (one with "$+\varrho$" and the other with "$-\varrho$") given by formula (4.6) for a fixed solution (α,β) of system (3.5). Indeed, it is sufficient to write explicitly a column of $B-A$. For simplicity, below we compute $(1+\alpha^2+\beta^2)(B_3 - A_3)$:

$$\begin{pmatrix} (1+\beta^2)x_3 - \alpha\beta y_3 - \alpha\frac{x_1y_2-y_1x_2}{\pm\varrho} \\ \\ -\alpha\beta x_3 + (1+\alpha^2)y_3 - \beta\frac{x_1y_2-y_1x_2}{\pm\varrho} \\ \\ \alpha x_3 + \beta y_3 + \frac{x_1y_2-y_1x_2}{\pm\varrho} \end{pmatrix} - \begin{pmatrix} (1+\alpha^2+\beta^2)x_3 \\ \\ (1+\alpha^2+\beta^2)y_3 \\ \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\alpha^2 x_3 - \alpha\beta y_3 - \alpha\frac{x_1y_2-y_1x_2}{\pm\varrho} \\ \\ -\alpha\beta x_3 - \beta^2 y_3 - \beta\frac{x_1y_2-y_1x_2}{\pm\varrho} \\ \\ \alpha x_3 + \beta y_3 + \frac{x_1y_2-y_1x_2}{\pm\varrho} \end{pmatrix} = \nu \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix}, \tag{4.7}$$

with $\nu = \alpha x_3 + \beta y_3 + \frac{x_1y_2-y_1x_2}{\pm\varrho}$. This means that the direction of projection is given by the column vector

$$U = \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix}, \tag{4.8}$$

and thus the two Pohlke matrices given by (4.6) are conjugate.

Remark 4.2 Let B the Pohlke matrix given by (4.6). Then, by (4.8), B corresponds to the orthogonal projection onto the plane $\{z=0\}$ iff $(\alpha,\beta)=(0,0)$ iff $(\gamma,\lambda)=(\frac{\pi}{2},1)$ iff A_1, A_2 are nonzero, orthogonal rows of equal norm, as we have already seen in Prop. 1.2. □

Multiplicity of Pohlke matrices.

Having the expression (4.6), it is clear that there are at most four different Pohlke's matrices. On the other hand, still from (4.6), taking into account Lemma 3.2, we see that for B_3 we have the following possibilities:

$$B_3 = \pm\frac{\alpha A_1 + \beta A_2}{1+\alpha^2+\beta^2} \pm \frac{A_1 \wedge A_2}{\varrho}, \tag{4.9}$$

where, now, (α,β) is a fixed solution of system (3.5). Thus, since A_1, A_2 are linearly independent, if $(\alpha,\beta)\neq(0,0)$ we find that

$$B_3 = \pm C \pm D, \tag{4.10}$$

with C, D two linearly independent rows. Hence, taking into account Prop. 1.2 and Lemma 3.2, in case of *non-orthogonal* projection there are exactly four distinct possibilities for B_3, i.e., four different Pohlke matrices. Besides, by (4.8), for every solution $(\alpha,\beta)\neq(0,0)$ of system (3.5) there are two distinct, conjugate Pohlke matrices. □

Examples of Pohlke matrices.

(1) Let us consider the matrix

$$A = \begin{pmatrix} \sqrt{14/3} & \sqrt{14/3} & \sqrt{14/3} \\ 1 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.11}$$

It is clear that $\|A_1\| = \|A_2\| = \sqrt{14}$ and $A_1 \perp A_2$. Thus, by Proposition 1.2, there are only two Pohlke matrices. Namely,

$$B = \begin{pmatrix} \sqrt{14/3} & \sqrt{14/3} & \sqrt{14/3} \\ 1 & 2 & -3 \\ -5/\sqrt{3} & 4/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \tag{4.12}$$

or

$$B = \begin{pmatrix} \sqrt{14/3} & \sqrt{14/3} & \sqrt{14/3} \\ 1 & 2 & -3 \\ 5/\sqrt{3} & -4/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}. \tag{4.13}$$

(2) Let us consider the matrix

$$A = \begin{pmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & \sqrt{2} \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.14}$$

The hypotheses of Proposition 1.2 are not verified. We have

$$\begin{aligned} \lambda &= \frac{\|A_1\|}{\|A_2\|} = \frac{2}{\sqrt{2}} = \sqrt{2}, \\ \cos\gamma &= \frac{A_1 \cdot A_2}{\|A_1\|\,\|A_2\|} = \frac{\sqrt{3}}{2\sqrt{2}}, \end{aligned} \tag{4.15}$$

and then

$$\eta = \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma} = 2, \qquad \varrho = \frac{\|A_2\|}{\sqrt{\eta}} = \frac{\sqrt{2}}{\sqrt{2}} = 1. \tag{4.16}$$

Thus, applying Lemma 3.2, we find

$$(\alpha, \beta) = \pm\left(\sqrt{\eta\lambda^2 - 1}\,, \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\,\right) = \pm\left(\sqrt{3}\,, 1\right). \tag{4.17}$$

With these two solutions for (α, β) we can determine $\widetilde{\mathcal{G}}^{-1}$. We have:

$$\widetilde{\mathcal{G}}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -\sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & 4 & -1 \\ \sqrt{3} & 1 & 1 \end{pmatrix} \quad \text{(i)} \tag{4.18}$$

or

$$\widetilde{\mathcal{G}}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -\sqrt{3} & \sqrt{3} \\ -\sqrt{3} & 4 & 1 \\ -\sqrt{3} & -1 & 1 \end{pmatrix} \quad \text{(ii)} \tag{4.19}$$

Having $A_1 \wedge A_2 = \left(-\sqrt{2}\,, \sqrt{2}\,, -1\right)$ and $\varrho = 1$, for the matrix $\mathcal{G}$ we find:

$$\mathcal{G} = \begin{pmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & \sqrt{2} \\ 1 & 1 & 0 \\ -\sqrt{2} & \sqrt{2} & -1 \end{pmatrix} \quad \text{(I)} \tag{4.20}$$

or

$$\mathcal{G} = \begin{pmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & \sqrt{2} \\ 1 & 1 & 0 \\ \sqrt{2} & -\sqrt{2} & 1 \end{pmatrix} \quad \text{(II)} \tag{4.21}$$

Combining the matrices (4.18), (4.19), (4.20), (4.21) we have four possibilities, say $B_{(i,I)}$, $B_{(i,II)}$, $B_{(ii,I)}$, $B_{(ii,II)}$, for the Pohlke matrix $B = \varrho\,\widetilde{\mathcal{G}}^{-1}\,\mathcal{G}$. More precisely, we have:

$$B_{(i,I)} = \frac{1}{5}\begin{pmatrix} \sqrt{6}-1 & 1-\sqrt{6} & 2\sqrt{2}+\sqrt{3} \\ \frac{5+2\sqrt{2}+\sqrt{3}}{2} & \frac{5-2\sqrt{2}-\sqrt{3}}{2} & 1-\sqrt{6} \\ \frac{5-2\sqrt{2}-\sqrt{3}}{2} & \frac{5+2\sqrt{2}+\sqrt{3}}{2} & \sqrt{6}-1 \end{pmatrix}, \tag{4.22}$$

$$B_{(i,II)} = \frac{1}{5}\begin{pmatrix} -\sqrt{6}-1 & \sqrt{6}+1 & 2\sqrt{2}-\sqrt{3} \\ \frac{5-2\sqrt{2}+\sqrt{3}}{2} & \frac{5+2\sqrt{2}-\sqrt{3}}{2} & -\sqrt{6}-1 \\ \frac{5+2\sqrt{2}-\sqrt{3}}{2} & \frac{5-2\sqrt{2}+\sqrt{3}}{2} & \sqrt{6}+1 \end{pmatrix}, \tag{4.23}$$

$$B_{(ii,I)} = \frac{1}{5}\begin{pmatrix} -\sqrt{6}-1 & \sqrt{6}+1 & 2\sqrt{2}-\sqrt{3} \\ \frac{5-2\sqrt{2}+\sqrt{3}}{2} & \frac{5+2\sqrt{2}-\sqrt{3}}{2} & -\sqrt{6}-1 \\ \frac{-5-2\sqrt{2}+\sqrt{3}}{2} & \frac{-5+2\sqrt{2}-\sqrt{3}}{2} & -\sqrt{6}-1 \end{pmatrix}, \tag{4.24}$$

$$B_{(ii,II)} = \frac{1}{5}\begin{pmatrix} \sqrt{6}-1 & 1-\sqrt{6} & 2\sqrt{2}+\sqrt{3} \\ \frac{5+2\sqrt{2}+\sqrt{3}}{2} & \frac{5-2\sqrt{2}-\sqrt{3}}{2} & 1-\sqrt{6} \\ \frac{-5+2\sqrt{2}+\sqrt{3}}{2} & \frac{-5-2\sqrt{2}-\sqrt{3}}{2} & 1-\sqrt{6} \end{pmatrix}. \tag{4.25}$$

Note that the couples of conjugate Pohlke matrices are $B_{(i,I)}$, $B_{(i,II)}$ and $B_{(ii,I)}$, $B_{(ii,II)}$ and the corresponding directions of projection are:

$$U_{(i)} = \begin{pmatrix} -\sqrt{3} \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad U_{(ii)} = \begin{pmatrix} \sqrt{3} \\ 1 \\ 1 \end{pmatrix}. \tag{4.26}$$

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