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ON POLHKE'S TYPE PROJECTIONS IN THE CYLINDRICAL CASE

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ABSTRACT. Given three non-parallel segments OP_1, OP_2, OP_3 in a plane ω, we consider the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ having as conjugate semi-diameters the pairs (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1), respectively. We find the necessary and sufficient conditions for (i) the existence of a common point $P \in \mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1}$ and (ii) the existence of a pair of parallel and distinct lines tangent to the three ellipses. In this later case, we solve the problem by introducing the definition of cylindrical Pohlke's projection.

1. INTRODUCTION AND MOTIVATIONS

Given two non-parallel segments OP_1, OP_2 in a plane ω, we set

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{for} \quad h, k \neq 0 \tag{1.1}$$

and then we consider the three concentric ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ determined by the pairs of conjugate semi-diameter (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1), respectively. It is possible to show that there exist at most two distinct ellipses, with center O, which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$. More precisely,

1) the *Pohlke's ellipse* $\mathcal{E}_{\mathrm{P}}$, which exists for every choice of $h, k \neq 0$ (see [1, 2, 3]). A pair of conjugate semi-diameters of $\mathcal{E}_{\mathrm{P}}$ is given by the vectors (see [4, 6]):

$$\frac{k\overrightarrow{OP_1} - h\overrightarrow{OP_2}}{\sqrt{h^2+k^2}} \quad \text{and} \quad \sqrt{\frac{1+h^2+k^2}{h^2+k^2}}\,\overrightarrow{OP_3}. \tag{1.2}$$

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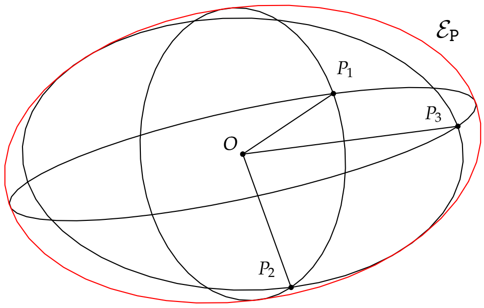

Figure 1. Pohlke's ellipse with $P_1 = (1.5, 1)$, $P_2(0.8, -2.2)$, $\overrightarrow{OP_3} = 2\overrightarrow{OP_1} + 0.7\overrightarrow{OP_2}$.

2) the *secondary Pohlke's ellipse* $\mathcal{E}_{\mathrm{S}}$ (see [9, 5, 6]). For $h, k \neq 0$, it exists if and only if

$$g(h,k) \overset{\text{def}}{=} (h+k+1)(h+k-1)(h-k+1)(h-k-1) > 0. \tag{1.3}$$

The area of $\mathcal{E}_{\mathrm{S}}$ is always strictly greater than that of $\mathcal{E}_{\mathrm{P}}$ and a pair of conjugate semi-diameters of $\mathcal{E}_{\mathrm{S}}$ is given by the vectors ([5, 6]):

$$\frac{K\overrightarrow{OP_1} - H\overrightarrow{OP_2}}{\sqrt{H^2+K^2}} \quad \text{and} \quad \sqrt{\frac{g + H^2 + K^2}{g(H^2+K^2)}}\,\Big(H\overrightarrow{OP_1} + K\overrightarrow{OP_2}\Big), \tag{1.4}$$

where

$$H(h,k) \overset{\text{def}}{=} h(h^2 - k^2 - 1), \quad K(h,k) \overset{\text{def}}{=} k(h^2 - k^2 + 1). \tag{1.5}$$

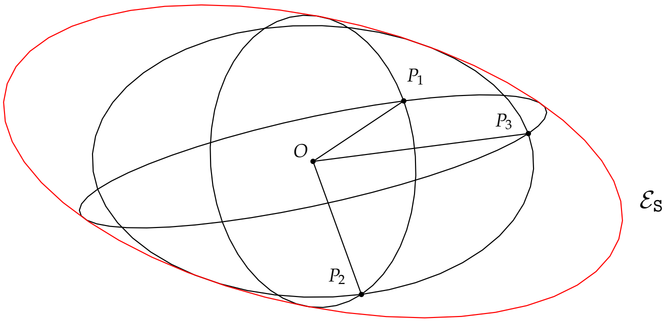

Figure 2. Secondary Pohlke's ellipse with $P_1 = (1.5, 1)$, $P_2(0.8, -2.2)$, $\overrightarrow{OP_3} = 2\overrightarrow{OP_1} + 0.7\overrightarrow{OP_2}$.

When instead of (1.3) we have $g(h,k) < 0$,[1] two other possibilities arises (hyperbolic Pohlke's conics) depending on wether the quantity

$$g + H^2 + K^2 \equiv (h^2 + k^2 - 1)\left[(h^2 - k^2)^2 - 1\right] \tag{1.6}$$

is negative or positive. In [7, 8] it was proved that:

3) there exists at most one concentric ellipse $\mathcal{E}_{\mathrm{I}}$ inscribed in $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ and it exists if and only if $g(h,k) < 0$ and $g + H^2 + K^2 < 0$. A pair of conjugate

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[1] Note that $g(h,k) < 0 \Rightarrow h, k \neq 0$.

semi-diameters is given, as for $\mathcal{E}_{\mathrm{S}}$, by the expressions

$$\frac{K\overrightarrow{OP_1}-H\overrightarrow{OP_2}}{\sqrt{H^2+K^2}} \quad \text{and} \quad \sqrt{\frac{g+H^2+K^2}{g(H^2+K^2)}}\left(H\overrightarrow{OP_1}+K\overrightarrow{OP_2}\right); \tag{1.7}$$

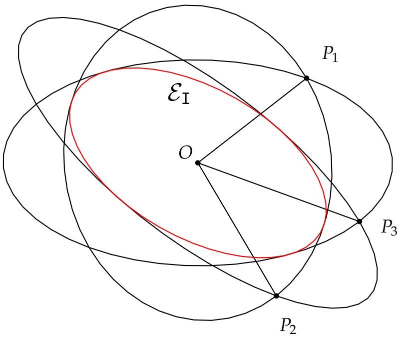

Figure 3. Inscribed ellipse $\mathcal{E}_{\mathrm{I}}$ with $P_1 = (1.8, 1.4)$, $P_2(1.3, -2.2)$, $\overrightarrow{OP_3} = 0.8\overrightarrow{OP_1} + 0.95\overrightarrow{OP_2}$.

4) there exists at most one concentric hyperbola $\mathcal{H}_{\mathrm{C}}$ which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ and it exists if and only if $g(h,k) < 0$ and $g + H^2 + K^2 > 0$. A pair of transverse (Σ_{tr}) and "imaginary" (Σ_{im}) conjugate semi-diameters is given by the vectors

$$\overrightarrow{\Sigma}_{tr} = \frac{K\overrightarrow{OP_1}-H\overrightarrow{OP_2}}{\sqrt{H^2+K^2}} \quad \text{and} \quad \overrightarrow{\Sigma}_{im} = \sqrt{-\frac{g+H^2+K^2}{g(H^2+K^2)}}\left(H\overrightarrow{OP_1}+K\overrightarrow{OP_2}\right), \tag{1.8}$$

respectively. [2]

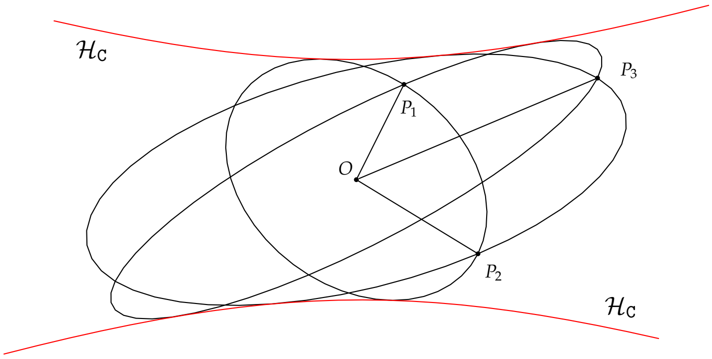

Figure 4. Hyperbola $\mathcal{H}_{\mathrm{C}}$ with $P_1 = (0.9, 1.8)$, $P_2(2.3, -1.4)$, $\overrightarrow{OP_3} = 2\overrightarrow{OP_1} + 1.2\overrightarrow{OP_2}$.

2. MAIN RESULTS

Here we investigate the residual cases that are not covered by the previous results:

(i) $g(h,k) < 0$ and $(h^2+k^2-1)\left[(h^2-k^2)^2-1\right] = 0$;

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[2] This terminology is not very common. In other words, we mean that a parametrization of $\mathcal{H}_{\mathrm{C}}$ is given by the expression $P(t) = O \pm \cosh t\,\overrightarrow{\Sigma}_{tr} + \sinh t\,\overrightarrow{\Sigma}_{im}$, for $t \in \mathbb{R}$.

(ii) $h, k \neq 0$ and $g(h,k) = 0$.

In case (i) we will show that, for $g(h,k) < 0$,

$$\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} \neq \emptyset \quad \text{if and only if} \quad (h^2 + k^2 - 1)\left[(h^2 - k^2)^2 - 1\right] = 0. \tag{2.1}$$

Furthermore, we have

$$\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} = \{\pm P_1\} \text{ or } \{\pm P_2\} \text{ or } \{\pm P_3\},^{3} \tag{2.2}$$

depending on whether $h^2 - k^2 - 1 = 0$ or $h^2 - k^2 + 1 = 0$ or $h^2 + k^2 - 1 = 0$ are respectively valid. So $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1}$ contains one and only one of points P_1, P_2, P_3.

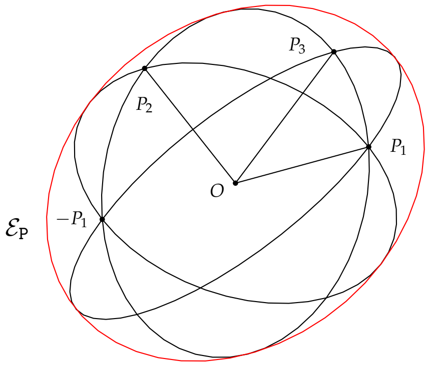

Figure 5. With $P_1 = (2.2, 0.6)$, $P_2(-1.5, 1.9)$ and $\overrightarrow{OP_3} = 1.25\overrightarrow{OP_1} + 0.75\overrightarrow{OP_2}$, $\mathcal{E}_{\mathrm{P}}$ and the intersection of $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$ at $\pm P_1$.

In case (ii) we will prove that there exists a unique a pair $\mathcal{T}_-$, $\mathcal{T}_+$ of distinct and parallel lines, tangent to $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$. More precisely, setting $\eta = \operatorname{sgn}\big(hk(h^2 + k^2 - 1)\big)$,[4] $\mathcal{T}_-$, $\mathcal{T}_+$ are the lines passing through the points

$$O - \frac{\overrightarrow{OP_1} - \eta\overrightarrow{OP_2}}{\sqrt{2}}, \quad O + \frac{\overrightarrow{OP_1} - \eta\overrightarrow{OP_2}}{\sqrt{2}} \tag{2.3}$$

respectively, and parallel to the vector

$$\overrightarrow{OP_1} + \eta\overrightarrow{OP_2}. \tag{2.4}$$

This is the result that one can expect observing in (1.4) (or (1.8)) the *limit behaviour* of the conjugate semi-diameters of the ellipse $\mathcal{E}_{\mathrm{S}}$ (or hyperbola $\mathcal{H}_{\mathrm{C}}$) as (h,k) tends in $\{g > 0\}$ (in $\{g < 0\}$) to a limit point $(\bar{h}, \bar{k})$ such that $\bar{h}, \bar{k} \neq 0$ and $g(\bar{h}, \bar{k}) = 0$.

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[3] With $-P$ we denotes the symmetric of P with respect to O; $\{\pm P\} = \{-P, P\}$.

[4] Here $\operatorname{sgn}(t) = 1$ for $t > 0$ and $\operatorname{sgn}(t) = -1$ for $t < 0$. Note that $h^2 + k^2 - 1 \neq 0$, if $h, k \neq 0$ and $g(h,k) = 0$.

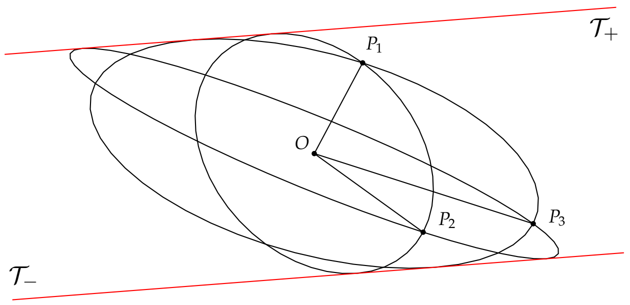

Figure 6. Tangent lines $\mathcal{T}_-$, $\mathcal{T}_+$ with $P_1 = (0.8, 1.5)$, $P_2(1.8, -1.3)$, $\overrightarrow{OP_3} = 0.7\overrightarrow{OP_1} + 1.7\overrightarrow{OP_2}$.

We will prove this fact by showing that the existence of the pair $\mathcal{T}_-$, $\mathcal{T}_+$ is equivalent to the existence of a *cylindrical* Pohlke's projection $\Pi : \mathbb{E}^3 \to \omega$. See Def. 4.5 below.

2.1. **Some degenerate cases.** We conclude this introduction recalling that, in some cases, is possible to make sense of $\mathcal{E}_{\text{P}}$, $\mathcal{E}_{\text{S}}$ and $\mathcal{H}_{\text{C}}$ when $OP_3 \parallel OP_1$ or $OP_3 \parallel OP_2$, i.e., in (1.1) h or k are zero. This can be done introducing degenerate ellipses with parallel conjugate semi-diameters ([1, 5]). For instance, if $A \neq O$ and $OA \parallel OB$, [5] the degenerate ellipse $\mathcal{E}_{A,B}$ is simply the segment $MN \parallel OA$ such that

$$|MN|^2 = 4\big(|OA|^2 + |OB|^2\big), \quad \frac{M+N}{2} = O, ^6 \tag{2.5}$$

and we say that $\mathcal{E}_{\text{P}}$ (or $\mathcal{E}_{\text{S}}$, $\mathcal{H}_{\text{C}}$) circumscribes $\mathcal{E}_{A,B}$ if $M, N \in \mathcal{E}_{\text{P}}$ (or $\mathcal{E}_{\text{S}}$, $\mathcal{H}_{\text{C}}$). With this argument it is possible to define the Pohlke's ellipse $\mathcal{E}_{\text{P}}$ even if both h, k are zero (expressions (1.2) remain valid if at leat one of h, k is non-zero). We can also make sense of $\mathcal{E}_{\text{S}}$ and $\mathcal{H}_{\text{C}}$ for $(h,k) = (\pm 1, 0)$ or $(h,k) = (0, \pm 1)$. But in these degenerate cases $\mathcal{E}_{\text{S}}$ and $\mathcal{H}_{\text{C}}$ are not unique and, clearly, the expressions (1.4), (1.8) are no longer valid. See [5, 6, 7, 8].

3. CASE (I): $g < 0$ AND $g + H^2 + K^2 = 0$

First we note that the condition (i) is equivalent to

$$h, k \neq 0 \quad \text{and} \quad (h^2 + k^2 - 1)\left[(h^2 - k^2)^2 - 1\right] = 0. \tag{3.1}$$

Then, we prove the following:

Theorem 3.1. *Suppose* $OP_1 \nparallel OP_2$ *and* (1.1). *Then,*

$$\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} = \{\pm P_3\} \quad \Leftrightarrow \quad h^2 + k^2 - 1 = 0; \tag{3.2}$$

$$\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} = \{\pm P_2\} \quad \Leftrightarrow \quad h^2 - k^2 + 1 = 0; \tag{3.3}$$

$$\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} = \{\pm P_1\} \quad \Leftrightarrow \quad h^2 - k^2 - 1 = 0. \tag{3.4}$$

Proof. Let us suppose $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} = \{\pm P_3\}$.
This means that $P_3 \in \mathcal{E}_{P_1,P_2}$. Noting that $\mathcal{E}_{P_1,P_2}$ can be defined by the parametric equation

$$P(t) = O + \cos t\,\overrightarrow{OP_1} + \sin t\,\overrightarrow{OP_2}, \quad t \in [0, 2\pi), \tag{3.5}$$

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[5] In particular, if OB is a null segment, that is $B = O$.

[6] Given P and Q, with $\frac{P+Q}{2}$ we indicate the midpoint of the segment PQ.

it is clear that

$$\overrightarrow{OP_3} = \cos\bar{t}\,\overrightarrow{OP_1} + \sin\bar{t}\,\overrightarrow{OP_2}, \tag{3.6}$$

for some $\bar{t} \in [0, 2\pi)$. Since $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are linearly independent, this gives

$$h = \cos\bar{t}, \quad k = \sin\bar{t}. \tag{3.7}$$

Conversely, let us suppose $h^2 + k^2 - 1 = 0$. It is clear that (3.7) holds for a unique $\bar{t} \in [0, 2\pi)$. Hence, taking into account the parametric equation (3.5), it follows that $\pm P_3 \in \mathcal{E}_{P_1,P_2}$ and than that $\{\pm P_3\} \subset \mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1}$.

To prove that there are no other points in $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1}$ it is enough to observe that $\{\pm P_3\} \subset \mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1}$ implies:

- $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} \subset \mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} = \{\pm P_2, \pm P_3\}$, because $P_2 \neq \pm P_3$; [7]
- $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3} \cap \mathcal{E}_{P_3,P_1} \subset \mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_3,P_1} = \{\pm P_1, \pm P_3\}$, because $P_1 \neq \pm P_3$. [7]

It is now sufficient to observe that $P_1 \neq \pm P_2$.

So far we have proved the equivalence (3.2). The proofs of (3.3) and (3.4) are similar, because it is sufficient to exchange the roles of P_1, P_2, P_3. For instance, if $h^2 - k^2 + 1 = 0$, we write

$$\overrightarrow{OP_2} = -\frac{h}{k}\,\overrightarrow{OP_1} + \frac{1}{k}\,\overrightarrow{OP_3}, \tag{3.8}$$

with

$$\left(-\frac{h}{k}\right)^2 + \left(\frac{1}{k}\right)^2 - 1 = 0. \tag{3.9}$$

We therefore immediately find ourselves in the case (3.2). □

4. CASE (II): $h, k \neq 0$ AND $g = 0$

To deal with case (ii) we resort to the parallel projection of a suitable cylinder with the axis perpendicular to ω. To this end, in the Euclidean space $\mathbb{E}^3$ we fix from now on a Cartesian system of coordinates x, y, z with the corresponding orthonormal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$. We also assume that

$$\omega \overset{\text{def}}{=} \Big\{(x, y, z) \in \mathbb{R}^3 \,\big|\, z = 0\Big\}, \tag{4.1}$$

ant that $O \in \omega$ is the origin of the coordinates.

Definition 4.1. *Given a plane* π *and a non-zero vector* $\mathbf{w}$, $\mathbf{w} \nparallel \pi$, *we say that* P, Q *are obliquely symmetrical with respect to* π, *in the direction of* $\mathbf{w}$, *if* $PQ \parallel \mathbf{w}$ *and* $\frac{P+Q}{2} \in \pi$. [6]

Definition 4.2. *Given a non-zero vector* $\mathbf{v} \nparallel \mathbf{k}$, *that is*

$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} \quad \text{with} \quad l, m, n \in \mathbb{R},\ l^2 + m^2 > 0, \tag{4.2}$$

we denote with $\pi_{\mathbf{v}}$ *the plane*

$$\pi_{\mathbf{v}} :\ lx + my = 0. \tag{4.3}$$

We say that P, P' *are* $\pi_{\mathbf{v}}$*−symmetric if* P, P' *are obliquely symmetrical with respect to the plane* $\pi_{\mathbf{v}}$, *in the direction of* $\mathbf{v}$.

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[7] It is elementary that $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_2,P_3}$ contains, at most, four distinct points. The same goes for $\mathcal{E}_{P_1,P_2} \cap \mathcal{E}_{P_3,P_1}$.

For $\rho > 0$, we denote with $\mathscr{C} = \mathscr{C}(\rho)$ the cylinder

$$\mathscr{C}(\rho) \overset{\text{def}}{=} \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = \rho^2\}. \tag{4.4}$$

Furthermore, given a point $P \in \mathscr{C}$, we indicate with $T_{\mathscr{C}}(P)$ the tangent plane to $\mathscr{C}$ at P. Namely, if $P = P(x_P, y_P, z_P)$, the plane

$$T_{\mathscr{C}}(P): \ x_P x + y_P y = \rho^2. \tag{4.5}$$

Definition 4.3. *Given a non-zero vector* $\mathbf{v} \nparallel \omega$*, we denote with* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *the parallel projection onto* ω*, in the direction of* $\mathbf{v}$*. We say that* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *is non-degenerate for* $\mathscr{C}$ *(or simply non-degenerate) if we also have* $\mathbf{v} \nparallel \mathbf{k}$.

Definition 4.4. *Let* $\Pi_{\mathbf{v}}$ *and* $\Pi_{\mathbf{w}}$ *be two non-degenerate projections onto* ω*. We say that* $\Pi_{\mathbf{v}}$ *is equivalent to* $\Pi_{\mathbf{w}}$ *if and only if* $\pi_{\mathbf{v}} = \pi_{\mathbf{w}}$.

Noting that $\mathscr{C}$ is $\pi_{\mathbf{v}}$–symmetric if $\mathbf{v} \nparallel \mathbf{k}$ (see Claim 5.1), we give the following definition:

Definition 4.5. *Let* $OP_1, OP_2, OP_3 \subset \omega$ *be three segments which are not contained in a line. A non-degenerate parallel projection* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *is a cylindrical Pohlke's projection for* OP_1, OP_2, OP_3 *if there are a cylinder* $\mathscr{C} = \mathscr{C}(\rho)$*, for some* $\rho > 0$*, and three points* $Q_1, Q_2, Q_3 \in \mathscr{C}$ *such that*

$$\Pi_{\mathbf{v}}(Q_i) = P_i \quad (1 \leq i \leq 3), \tag{4.6}$$

$$OQ_1 \parallel T_{\mathscr{C}}(Q_2), \ OQ_2 \parallel T_{\mathscr{C}}(Q_3) \ \text{ and } \ OQ_3 \parallel T_{\mathscr{C}}(Q_1'), \tag{4.7}$$

where $Q_1' \in \mathscr{C}$ *is* $\pi_{\mathbf{v}}-$*symmetric to* Q_1 *in the sense of Def. 4.2 above.*

Remark 4.6. *Def. 4.5 is an adaptation to the cylindrical case of the secondary Pohlke's projection definition given in* [5, Def. 1.2]*. With condition* (4.7) *we require that the intersections of* $\mathscr{C}$ *with the planes passing through* O, Q_1, Q_2*, through* O, Q_2, Q_3 *and through* O, Q_3, Q_1' *are three ellipses having as conjugate semi-diameters the pairs* (OQ_1, OQ_2)*,* (OQ_2, OQ_3) *and* (OQ_3, OQ_1')*, respectively. See Claim 5.14 below.*

If the segments OP_1, OP_2, OP_3 are not parallel to each other, we can think (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1) as pairs of conjugate semi-diameters of three concentric ellipses in the plane ω.

Definition 4.7. *Given* $OP, OQ \subset \omega$*,* $OP \nparallel OQ$*, we denote with* $\mathcal{E}_{P,Q}$ *the ellipse with* OP, OQ *as conjugate semi-diameters.*

Then, considering the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$, we give the following definition:

Definition 4.8. *Suppose* OP_1, OP_2, OP_3 *are non-parallel. We say that* $\mathcal{T} = \mathcal{T}_- \cup \mathcal{T}_+$ *is a cylindrical Pohlke's conic for* OP_1, OP_2, OP_3 *if* $\mathcal{T}_-, \mathcal{T}_+ \subset \omega$ *are distinct and parallel lines, tangent to three ellipse* $\mathcal{E}_{P_1,P_2}$*,* $\mathcal{E}_{P_2,P_3}$ *and* $\mathcal{E}_{P_3,P_1}$*.*[8]

With the previous definitions, we have:

Theorem 4.9. *Suppose the segments* OP_1, OP_2, OP_3 *are non-parallel. Then the following three properties are equivalent:*

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[8] In other words, $\mathcal{T}$ is a degenerate conic formed by a pair of distinct, parallel lines $\mathcal{T}_-, \mathcal{T}_+$ which are symmetric with respect to the origin O, and tangent to $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

(1) *there is a cylindrical Pohlke's projection* $\Pi_{\mathbf{v}}$ *for* OP_1, OP_2, OP_3;

(2) *there is a cylindrical Pohlke's conic* $\mathcal{T} = \mathcal{T}_- \cup \mathcal{T}_+$ *for* OP_1, OP_2, OP_3;

(3) $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$ *with* $h, k \neq 0$ *satisfying the condition*

$$g(h,k) \overset{\text{def}}{=} (h+k+1)(h+k-1)(h-k+1)(h-k-1) = 0. \tag{4.8}$$

If the above conditions are true, then $\mathcal{T}$ *is unique and* $\mathcal{T}_-, \mathcal{T}_+$ *satisfy* (2.3), (2.4); $\Pi_{\mathbf{v}}$ *is unique up to equivalence in the sense of Def. 4.4 and* $\mathscr{C} = \mathscr{C}(\rho)$ *with* ρ *half the distance between the lines* $\mathcal{T}_-, \mathcal{T}_+$. *Besides, we have* $\mathcal{T} = \Pi_{\mathbf{v}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{v}})$.

5. SOME BASIC GEOMETRIC FACTS

Here we will state (and partly prove) some elementary facts regarding the cylinder $\mathscr{C} = \mathscr{C}(\rho)$ defined in (4.4). We start with some symmetry properties.

Claim 5.1. *Let* $\pi_{\mathbf{v}}$ *be the plane introduced in Def. 4.2. Then* $\mathscr{C}$ *is* $\pi_{\mathbf{v}}-$*symmetric.*

Proof. Indeed, let r be any line parallel to $\mathbf{v}$, that is,

$$r : \begin{cases} x = x_o + lt \\ y = y_o + mt \\ z = z_0 + nt \end{cases} \qquad (t \in \mathbb{R}), \text{ for a suitable } P(x_o, y_o, z_o). \tag{5.1}$$

Introducing the expressions (5.1) into the equation of $\mathscr{C}$, we see that the points of $r \cap \mathscr{C}$ are determined by the real solutions of

$$(l^2 + m^2)t^2 + 2(lx_o + my_o)t + x_o^2 + y_o^2 = \rho^2. \tag{5.2}$$

Since $l^2 + m^2 \neq 0$, equation (5.2) is of second degree with roots t_1, t_2 such that

$$\frac{t_1 + t_2}{2} = -\frac{lx_o + my_o}{l^2 + m^2}. \tag{5.3}$$

Now, if $P \in \mathscr{C}$, the solutions of (5.2) are

$$t_1 = 0 \quad \text{and} \quad t_2 = -2\,\frac{lx_o + my_o}{l^2 + m^2}. \tag{5.4}$$

Hence $r \cap \mathscr{C} = \{P(t_1), P(t_2)\}$ with $P(t_1) = P$ and $P(t_2)$ such that

$$\frac{P(t_1) + P(t_2)}{2} = P\left(\frac{t_1 + t_2}{2}\right) \in \pi_{\mathbf{v}}, \tag{5.5}$$

because of (5.1), (5.3). Thus $P(t_2) = P'$. □

Remark 5.2. *From the proof of Claim 5.1 one can also see that* r *is tangent to* $\mathscr{C}$ *at* P *iff* $P \in \mathscr{C} \cap \pi_{\mathbf{v}}$. *In fact, if* $P \in \mathscr{C}$, *we have* $t_1 = t_2 \Leftrightarrow lx_o + my_o = 0$.

Definition 5.3. *Given* $\mathbf{v} \nparallel \mathbf{k}$, *we indicate with* $\mathtt{S}_{\mathbf{v}}$ *the oblique symmetry with respect to* $\pi_{\mathbf{v}}$, *in the direction of* $\mathbf{v}$. *That is the map* $P(x,y,z) \xrightarrow{\mathtt{S}_{\mathbf{v}}} P'(x',y',z')$ *given by*

$$\mathtt{S}_{\mathbf{v}}(x,y,z) = (x - 2\lambda l, y - 2\lambda m, z - 2\lambda n) \quad \text{with} \quad \lambda = \frac{lx + my}{l^2 + m^2}. \quad \square \tag{5.6}$$

Then, we can observe that

Remark 5.4. *We can also get Claim 5.1 directly from the oblique symmetry* $\mathtt{S}_{\mathbf{v}}$ *introduced in Def. 5.3. Indeed, it is easy to see that* $\mathtt{S}_{\mathbf{v}}(P) \in \mathscr{C}$ *iff* $P \in \mathscr{C}$.

5.1. **The projection of $\mathscr{C}$ and $\mathscr{C} \cap \pi_{\mathbf{v}}$ into ω.** To continue we suppose $\mathbf{v} \nparallel \omega, \mathbf{k}$. Namely, we assume that

$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} \quad \text{with} \quad l^2 + m^2 > 0,\ n \neq 0. \tag{5.7}$$

We can therefore define the non-degenerate projection $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$.

Definition 5.5. *Let* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *be non-degenerate. With* $\mathscr{C} = \mathscr{C}(\rho)$*, we set*

$$\mathscr{T}_{\mathbf{v}} \overset{\text{def}}{=} \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi_{\mathbf{v}}). \tag{5.8}$$

It is elementary to see that

$$\mathscr{T}_{\mathbf{v}} = \mathscr{T}_{\mathbf{v}}^{-} \cup \mathscr{T}_{\mathbf{v}}^{+}, \tag{5.9}$$

where $\mathscr{T}_{\mathbf{v}}^{-}$, $\mathscr{T}_{\mathbf{v}}^{+}$ are the lines parallel to the vector $l\mathbf{i} + m\mathbf{j}$ and passing through the points

$$O - \rho \frac{m\mathbf{i} - l\mathbf{j}}{\sqrt{l^2 + m^2}} \quad \text{and} \quad O + \rho \frac{m\mathbf{i} - l\mathbf{j}}{\sqrt{l^2 + m^2}}, \tag{5.10}$$

respectively. We then give the following definition:

Definition 5.6. *Given an ellipse* $\mathcal{E} \subset \omega$ *with center* O*, we say that* $\mathscr{T}_{\mathbf{v}} = \mathscr{T}_{\mathbf{v}}^{-} \cup \mathscr{T}_{\mathbf{v}}^{+}$ *is tangent to* $\mathcal{E}$ *if the parallel lines* $\mathscr{T}_{\mathbf{v}}^{-}$ *and* $\mathscr{T}_{\mathbf{v}}^{+}$ *are both tangent to* $\mathcal{E}$.

Besides, we can note three other simple facts:

Claim 5.7. *Let* $\Pi_{\mathbf{v}}, \Pi_{\mathbf{w}} : \mathbb{R}^3 \to \omega$ *be non-degenerate and equivalent in the sense of Def. 4.4. Besides, let* $\mathscr{C} = \mathscr{C}(\rho)$ *for a fixed* $\rho > 0$*. Then* $\mathscr{T}_{\mathbf{v}} = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi_{\mathbf{v}}) = \Pi_{\mathbf{w}}(\mathscr{C} \cap \pi_{\mathbf{w}}) = \mathscr{T}_{\mathbf{w}}$.

Claim 5.8. *Let* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *be non-degenerate. Then* $\Pi_{\mathbf{v}}(\mathscr{C}) = \mathbf{int}(\mathscr{T}_{\mathbf{v}})$*, where* $\mathbf{int}(\mathscr{T}_{\mathbf{v}}) \subset \omega$ *is the strip between* $\mathscr{T}_{\mathbf{v}}^{-}$ *and* $\mathscr{T}_{\mathbf{v}}^{+}$.

Claim 5.9. *If* π *is a plane through* O *and* $\pi \nparallel \mathbf{k}$*, then* $\mathscr{C} \cap \pi$ *is an ellipse in* π *with center* O.

We can now prove the following:

Claim 5.10. *Let* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *be non-degenerate and let* $\mathscr{C} = \mathscr{C}(\rho)$ *with* $\rho > 0$.

1) *Let* $\mathcal{E} \subset \Pi_{\mathbf{v}}(\mathscr{C})$ *be an ellipse with center* O *and tangent to* $\mathscr{T}_{\mathbf{v}} = \mathscr{T}_{\mathbf{v}}^{-} \cup \mathscr{T}_{\mathbf{v}}^{+}$*. Then there are* $\pi_{\mathbf{v}}-$*symmetric planes* π, π' *through* O *such that* $\pi, \pi' \nparallel \mathbf{v}, \mathbf{k}$ *and*

$$\mathcal{E} = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi) = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi'). \tag{5.11}$$

If (OP_1, OP_2) *is a pair of conjugate semi-diameters of* $\mathcal{E}$ *then there are* Q_1, Q_1', Q_2, Q_2' *in* $\mathscr{C}$ *such that* $\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathscr{C} = \{Q_1, Q_1'\}$, $\Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathscr{C} = \{Q_2, Q_2'\}$ *and* (OQ_1, OQ_2), (OQ_1', OQ_2') *are pairs of conjugate semi-diameters of the ellipses* $\mathscr{C} \cap \pi$ *and* $\mathscr{C} \cap \pi'$*, respectively.*

2) *Conversely, if* π *is a plane through* O *such that* $\pi \nparallel \mathbf{v}, \mathbf{k}$ *then* $\mathcal{E} = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi)$ *is an ellipse with center* O *and tangent to* $\mathscr{T}_{\mathbf{v}}$.

Proof. 1) Let $\mathcal{E} \subset \Pi_{\mathbf{v}}(\mathscr{C})$ be an ellipse with center O and tangent to $\mathscr{T}_{\mathbf{v}}$ at X_1. Besides, let $X_2 \in \mathcal{E}$ such that $OX_1 \nparallel OX_2$ (i.e., $X_2 \in \mathcal{E} \setminus \mathscr{T}_{\mathbf{v}}$). Since we assume $\mathcal{E} \subset \Pi_{\mathbf{v}}(\mathscr{C})$, we have

$$X_1, X_2 \in \Pi_{\mathbf{v}}(\mathscr{C}). \tag{5.12}$$

Thus there are $Y_1 \in \mathscr{C} \cap \pi_{\mathbf{v}}$ and $Y_2 \in \mathscr{C}$ such that

$$\Pi_{\mathbf{v}}(Y_1) = X_1,\ \ \Pi_{\mathbf{v}}(Y_2) = X_2 \quad \text{and} \quad OY_1 \nparallel OY_2.\,\text{[9]} \tag{5.13}$$

To proceed, let π be the plane through the points O, Y_1, Y_2. It is clear that $\pi \nparallel \mathbf{v}$, otherwise we would have $OX_1 = \Pi_{\mathbf{v}}(OY_1) \parallel \Pi_{\mathbf{v}}(OY_2) = OX_2$. Hence the restriction

$$\Pi_{\mathbf{v}}\Big|_{\pi} : \pi \longrightarrow \omega \quad \text{defines an affine transformation.} \tag{5.14}$$

Also note that $\pi \nparallel \mathbf{k}$.[10] Hence, by Claim 5.9, $\mathscr{C} \cap \pi$ is an ellipse with center O. By (5.14), the same goes for $\mathcal{Q} \overset{\text{def}}{=} \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi)$. Furthermore, by Claim 5.8,

$$X_1 \in \mathcal{Q} \quad \text{and} \quad \mathcal{Q} \subset \Pi_{\mathbf{v}}(\mathscr{C}) \quad \Longrightarrow \quad \mathcal{Q} \text{ is tangent to } \mathscr{T}_{\mathbf{v}} \text{ at } X_1.\text{[11]}$$

This means that $\mathcal{Q}$ has in common with $\mathcal{E}$ the point X_1, the tangent at X_1 and a second point X_2 such that $OX_1 \nparallel OX_2$. Since $\mathcal{E}$ and $\mathcal{Q}$ both have center at O, it follows that $\mathcal{E} = \mathcal{Q} = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi)$. Recalling also that $\mathscr{C}$ is $\pi_{\mathbf{v}}-$symmetric, if π' is $\pi_{\mathbf{v}}-$symmetric to π we immediately get

$$\Pi_{\mathbf{v}}(\mathscr{C} \cap \pi') = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi) = \mathcal{E}. \tag{5.15}$$

So (5.11) is verified and, in particular, this implies that $\pi' \nparallel \mathbf{v}, \mathbf{k}$.

Next, let OP_1, OP_2 be conjugate semi-diameters of $\mathcal{E}$. Having $\pi, \pi' \nparallel \mathbf{v}$, the restrictions

$$\Pi_{\mathbf{v}}\Big|_{\pi} : \pi \longrightarrow \omega \quad \text{and} \quad \Pi_{\mathbf{v}}\Big|_{\pi'} : \pi' \longrightarrow \omega \quad \text{are affine transformations.} \tag{5.16}$$

Then, by (5.15) and (5.16), there are $Q_1, Q_2 \in \mathscr{C} \cap \pi$ and $\tilde{Q}_1, \tilde{Q}_2 \in \mathscr{C} \cap \pi'$ such that

$$\Pi_{\mathbf{v}}(Q_1) = \Pi_{\mathbf{v}}(\tilde{Q}_1) = P_1, \quad \Pi_{\mathbf{v}}(Q_2) = \Pi_{\mathbf{v}}(\tilde{Q}_2) = P_2.$$

The pairs (OQ_1, OQ_2) and $(O\tilde{Q}_1, O\tilde{Q}_2)$ are therefore conjugate semi-diameters of the conics $\mathscr{C} \cap \pi$ and $\mathscr{C} \cap \pi'$, respectively. On the other hand, it easy to show that Q_i and $\tilde{Q}_i$ are necessarily $\pi_{\mathbf{v}}-$symmetric, that is, $\tilde{Q}_i = Q_i'$ and

$$\Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathscr{C} = \{Q_i, \tilde{Q}_i\} \quad \text{for } i = 1, 2.$$

2) Conversely, let π be a plane through the origin O such that $\pi \nparallel \mathbf{v}$ and $\pi \nparallel \mathbf{k}$. By Claim 5.9, $\mathscr{C} \cap \pi$ is an ellipse with center O and since we suppose $\pi \nparallel \mathbf{v}$, it is clear that (5.14) holds. Thus $\mathcal{E} = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi)$ is an ellipse in ω, with center O. Moreover, $\mathcal{E} \cap \mathscr{T}_{\mathbf{v}} \neq \emptyset$ because

$$(\mathscr{C} \cap \pi) \cap (\mathscr{C} \cap \pi_{\mathbf{v}}) = \mathscr{C} \cap (\pi \cap \pi_{\mathbf{v}}) \neq \emptyset.\,\text{[12]}$$

Taking into account that $\mathcal{E} \subset \Pi_{\mathbf{v}}(\mathscr{C})$, from Claim 5.8 we then deduce that $\mathcal{E}$ and $\mathscr{T}_{\mathbf{v}}$ are tangent at the points of $\Pi_{\mathbf{v}}\big(\mathscr{C} \cap \pi_{\mathbf{v}} \cap \pi\big)$. ☐

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[9] Note that Y_1 is unique. In fact, since $X_1 \in \mathscr{T}_{\mathbf{v}}$, the line through X_1 and parallel to $\mathbf{v}$ is tangent to $\mathscr{C}$ at a point of $\mathscr{C} \cap \pi_{\mathbf{v}}$. On the contrary Y_2 is not unique because $X_2 \notin \mathscr{T}_{\mathbf{v}}$. $\Pi_{\mathbf{v}}^{-1}(X_2) \cap \mathscr{C} = \{Y_2, Y_2'\}$ with Y_2, Y_2' $\pi_{\mathbf{v}}-$symmetric and $Y_2 \neq Y_2'$. See Claim 5.1 and Rem. 5.2 above.

[10] Since $O, Y_1 \in \pi_{\mathbf{v}} \cap \pi$, we have that $\pi \parallel \mathbf{k} \Rightarrow \pi = \pi_{\mathbf{v}}$. So $X_2 \in \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi) = \Pi_{\mathbf{v}}(\mathscr{C} \cap \pi_{\mathbf{v}}) = \mathscr{T}_{\mathbf{v}}$, which is not true because $OX_2 \nparallel OX_1$.

[11] Indeed, let t_1 be the tangent of $\mathcal{Q}$ at X_1. Since $X_1 \in \mathscr{T}_{\mathbf{v}}$, if $t_1 \nparallel \mathscr{T}_{\mathbf{v}}$ then $\mathcal{Q} \not\subset \mathbf{int}(\mathscr{T}_{\mathbf{v}})$. This fact contradicts Claim 5.8, because $\mathcal{Q} \subset \Pi_{\mathbf{v}}(\mathscr{C})$.

[12] Since $\pi \nparallel \mathbf{k}$, $\ell = \pi \cap \pi_{\mathbf{v}}$ is a straight line passing through O and not parallel to $\mathbf{k}$.

5.2. **Some properties of the tangent planes of $\mathscr{C}$.** To proceed, we recall that $T_{\mathscr{C}}(P)$ denotes the tangent plane to $\mathscr{C}$ at P. More precisely, if $\mathscr{C} = \mathscr{C}(\rho)$ and $P = P(x_P, y_P, z_P) \in \mathscr{C}$, $T_{\mathscr{C}}(P)$ is the plane defined by equation (4.5).

Claim 5.11. *If $P, Q \in \mathscr{C}$ and O is the origin of coordinates, then*

$$OP \parallel T_{\mathscr{C}}(Q) \quad \Leftrightarrow \quad OQ \parallel T_{\mathscr{C}}(P). \tag{5.17}$$

Proof. In fact, given $P = P(x_P, y_P, z_P) \in \mathscr{C}$ and $Q = Q(x_Q, y_Q, z_Q)$, we have that

$$OQ \parallel T_{\mathscr{C}}(P) \quad \Leftrightarrow \quad x_P x_Q + y_P y_Q = 0. \qquad \square \tag{5.18}$$

Noting that $\mathscr{C}$ is $\pi_{\mathbf{v}}-$symmetric, applying Claim 5.11 we easily obtain the following:

Claim 5.12. *If $P, Q \in \mathscr{C}$ and P', Q' are $\pi_{\mathbf{v}}-$symmetric to P, Q respectively, then*

$$OP \parallel T_{\mathscr{C}}(Q) \quad \Leftrightarrow \quad OP' \parallel T_{\mathscr{C}}(Q') \tag{5.19}$$

and

$$OP \parallel T_{\mathscr{C}}(Q') \quad \Leftrightarrow \quad OQ \parallel T_{\mathscr{C}}(P'). \tag{5.20}$$

Proof. Recalling Def. 5.3 and Rem. 5.4, we easily have

$$\mathtt{S}_{\mathbf{v}}(T_{\mathscr{C}}(Q)) = T_{\mathscr{C}}(Q'), \tag{5.21}$$

where $\mathtt{S}_{\mathbf{v}}$ is the oblique symmetry with respect to the plane $\pi_{\mathbf{v}}$, in the direction of $\mathbf{v}$. This immediately gives (5.19). Then (5.20) follows from (5.19) and Claim 5.11. $\square$

Definition 5.13. *Assuming $OP \nparallel OQ$, we denote with $\langle O, P, Q\rangle$ the plane through the origin O and the points P, Q. With $\mathcal{C}(P, Q)$ we indicate the conic*

$$\mathcal{C}(P, Q) \overset{\text{def}}{=} \mathscr{C} \cap \langle O, P, Q\rangle. \tag{5.22}$$

Moreover, given $R \in \mathcal{C}(P, Q)$, we will denote with $T_{\mathcal{C}(P,Q)}(R) \subset \langle O, P, Q\rangle$ the tangent line to $\mathcal{C}(P, Q)$ passing through the point R.

Claim 5.14. *Suppose $P, Q \in \mathscr{C}$. Then $OP \parallel T_{\mathscr{C}}(Q) \Leftrightarrow OP \nparallel OQ$ and $\mathcal{C}(P, Q) = \mathscr{C} \cap \langle O, P, Q\rangle$ is an ellipse with (OP, OQ) as a pair of conjugate semi-diameters.*

Proof. $\Rightarrow$ By (5.18) we immediately have that

$$P, Q \in \mathscr{C} \ \text{ and } \ OP \parallel T_{\mathscr{C}}(Q) \quad \Rightarrow \quad OP \nparallel OQ. \tag{5.23}$$

Besides, from $OP \parallel T_{\mathscr{C}}(Q)$ it also follows that $\langle O, P, Q\rangle \nparallel \mathbf{k}$. Therefore $\mathcal{C}(P, Q) = \mathscr{C} \cap \langle O, P, Q\rangle$ is an ellipse in $\langle O, P, Q\rangle$, with center O. Noting that $T_{\mathscr{C}}(Q) \nparallel \langle O, P, Q\rangle$, we deduce that the tangent line $T_{\mathcal{C}(P,Q)}(Q)$ satisfies

$$T_{\mathcal{C}(P,Q)}(Q) = T_{\mathscr{C}}(Q) \ \cap \langle O, P, Q\rangle, \tag{5.24}$$

because it is clear that $T_{\mathcal{C}(P,Q)}(Q) \subset T_{\mathscr{C}}(Q)$ and that $T_{\mathcal{C}(P,Q)}(Q) \subset \langle O, P, Q\rangle$.
Then, since $OP \parallel \langle O, P, Q\rangle$ and we suppose $OP \parallel T_{\mathscr{C}}(Q)$, it follows that

$$OP \parallel T_{\mathcal{C}(P,Q)}(Q). \tag{5.25}$$

Moreover, by Claim 5.11, $OP \parallel T_{\mathscr{C}}(Q) \ \Leftrightarrow \ OQ \parallel T_{\mathscr{C}}(P)$. So with the same arguments used above we can prove that

$$OQ \parallel T_{\mathcal{C}(P,Q)}(P). \tag{5.26}$$

Since we know that $\mathcal{C}(P,Q)$ is an ellipse, from (5.25) and (5.26) we deduce that (OP, OQ) is a pair of conjugate semi-diameters of $\mathcal{C}(P,Q)$.
$\Leftarrow$ It follows from the properties of semi-diameters, because $T_{\mathcal{C}(P,Q)}(Q) \subset T_{\mathscr{C}}(Q)$. $\square$

Subsequently, taking into account Def. 4.7, we have:

Claim 5.15. *Let* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *be a parallel projection and let* $Q_1, Q_2 \in \mathscr{C}$ *such that* $OQ_1 \parallel T_{\mathscr{C}}(Q_2)$. *Let* $P_1 = \Pi_{\mathbf{v}}(Q_1)$, $P_2 = \Pi_{\mathbf{v}}(Q_2)$. *If* $OP_1 \nparallel OP_2$, *then* $\Pi_{\mathbf{v}}\Big|_{\langle O,Q_1,Q_2\rangle} : \langle O, Q_1, Q_2\rangle \to \omega$ *defines an affine map such that*

$$\Pi_{\mathbf{v}}(\mathcal{C}(Q_1, Q_2)) = \mathcal{E}_{P_1,P_2}. \tag{5.27}$$

If we further suppose that $\Pi_{\mathbf{v}}$ *is non-degenerate, then* $\mathcal{E}_{P_1,P_2}$ *is tangent to* $\mathscr{T}_{\mathbf{v}}$.

Proof. By Claim 5.14, we already know that $OQ_1 \nparallel OQ_2$ and that $\mathcal{C}(Q_1, Q_2)$ is an ellipse with conjugate semi-diameters OQ_1,OQ_2. Now, assuming $OP_1 \nparallel OP_2$, we have that

$$OP_1 \nparallel OP_2 \quad \text{and} \quad \Pi_{\mathbf{v}}(Q_1) = P_1,\ \Pi_{\mathbf{v}}(Q_2) = P_2 \quad \Longrightarrow \quad \mathbf{v} \nparallel \langle O, Q_1, Q_2\rangle. \tag{5.28}$$

So, the restriction

$$\Pi_{\mathbf{v}}\Big|_{\langle O,Q_1,Q_2\rangle} : \langle O, Q_1, Q_2\rangle \to \omega$$

defines an affine transformation. Having $\Pi_{\mathbf{v}}(OQ_1) = OP_1$ and $\Pi_{\mathbf{v}}(OQ_2) = OP_2$, it is therefore clear that (5.27) holds. Finally, if $\Pi_{\mathbf{v}}$ is also non-degenerate, i.e., $\mathbf{v} \nparallel \mathbf{k}$, from part 2) of Claim 5.10 we immediately have that $\mathcal{E}_{P_1,P_2}$ is tangent to $\mathscr{T}_{\mathbf{v}}$. $\square$

Finally, we note the following fact:

Remark 5.16. *Let* $\mathtt{S}_{\mathbf{v}}$ *be the oblique symmetry with respect to* $\pi_{\mathbf{v}}$ *given by* (5.6).
If $Q_1, Q_2, Q_3 \in \mathscr{C}$ *satisfy the conditions* (4.6), (4.7) *of Def. 4.5 then, by Claim 5.12, also the points* $Q'_1 = \mathtt{S}_{\mathbf{v}}(Q_1)$, $Q'_2 = \mathtt{S}_{\mathbf{v}}(Q_2)$ *and* $Q'_3 = \mathtt{S}_{\mathbf{v}}(Q_3)$ *satisfy* (4.6), (4.7). *This means that in Def. 4.5 the triads* Q_1, Q_2, Q_3 *and* Q'_1, Q'_2, Q'_3 *are perfectly equivalent.*

6. CYLINDRICAL POHLKE'S PROJECTION IN THE CIRCULAR CASE

To go further, let us now determine the cylindrical Pohlke's projections in the *circular case*. More precisely, instead of three generic non-parallel segments OP_1, OP_2, OP_3 we take three non-parallel segments ON_1,ON_2,$ON_3 \subset \omega$ such that

$$ON_1 \perp ON_2 \quad \text{and} \quad |ON_1| = |ON_2| = 1. \tag{6.1}$$

To avoid any confusion between the *circular case* and the *general case*, in the following we also use R_1, R_2 and R_3 instead of Q_1, Q_2 and Q_3, respectively.

To begin with, according to Def. 4.5, we need to find $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ non-degenerate and then $R_1, R_2 \in \mathscr{C}(\rho)$ such that

$$\Pi_{\mathbf{v}}(R_1) = N_1, \ \ \Pi_{\mathbf{v}}(R_2) = N_2 \ \text{ with } \ OR_1 \parallel T_{\mathscr{C}}(R_2).$$

Assuming such a projection exists, from Claim 5.15 we deduce that $\mathcal{E}_{N_1,N_2}$ must be tangent to $\mathscr{T}_{\mathbf{v}}$. Since $\mathscr{E}_{N_1,N_2}$ is the circle with center O and unit radius, we have:

Claim 6.1. *If* (6.1) *holds and if there is a cylindrical Pohlke's projection for* ON_1, ON_2, ON_3 *(according to Def. 4.5), then* $\rho = 1$. *That is, we have*

$$\mathscr{C} = \mathscr{C}(1) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}. \tag{6.2}$$

After this, again assuming that the cylindrical Pohlke's projection $\Pi_{\mathbf{v}}$ exists, we note that (6.1), (6.2) imply $N_1, N_2 \in \mathscr{C}$. Thus we must have:

$$N_1 = R_1 \text{ or } R_1' \quad \text{and} \quad N_2 = R_2 \text{ or } R_2'.^{13} \tag{6.3}$$

But to satisfy the conditions of Def. 4.5 it is necessary to set

$$R_1 = N_1 \text{ and } R_2 = N_2 \tag{6.4}$$

or, equivalently by Rem. 5.16, $R_1' = N_1$ and $R_2' = N_2$.[14]
In fact, if we set $R_1 = N_1$ and $R_2' = N_2$, applying Claim 5.12, we find:

$$OR_3 \parallel T_{\mathscr{C}}(R_1') \Leftrightarrow OR_1 \parallel T_{\mathscr{C}}(R_3') \Leftrightarrow ON_1 \parallel T_{\mathscr{C}}(R_3'), \tag{6.5}$$

$$OR_2 \parallel T_{\mathscr{C}}(R_3) \Leftrightarrow OR_2' \parallel T_{\mathscr{C}}(R_3') \Leftrightarrow ON_2 \parallel T_{\mathscr{C}}(R_3'). \tag{6.6}$$

Now, from (5.18), it is easy to see that

$$ON_1 \parallel T_{\mathscr{C}}(R_3') \text{ and } ON_2 \parallel T_{\mathscr{C}}(R_3') \implies OR_3' \perp \omega^{15} \tag{6.7}$$

and the latter condition cannot be satisfied if $R_3' \in \mathscr{C}$. Since the same argument works if we try to define $R_1' = N_1$ and $R_2 = N_2$, we are forced to assume (6.4) (or, equivalently, $R_1' = N_1$ and $R_2' = N_2$). Moreover, by choosing $R_1 = N_1$ and $R_2 = N_2$, we also find

$$R_3 \neq R_3'. \tag{6.8}$$

Indeed, if $R_3 = R_3'$, from Claim 5.12 and condition (4.7) we easily deduce that $ON_1 \parallel T_{\mathscr{C}}(R_3)$ and $ON_2 \parallel T_{\mathscr{C}}(R_3)$. Hence, as in (6.7), we find $OR_3 \perp \omega$ which cannot be satisfied. In conclusion, noting that (6.8) implies $R_3R_3' \parallel \mathbf{v}$, we can say that:

Conditions 6.2. *Having fixed the points* $R_1 = N_1$, $R_2 = N_2$ *as in* (6.4), *to have a cylindrical Pohlke's projection for* ON_1, ON_2, ON_3 *as in* (6.1), *it is necessary and sufficient to determine* $R_3, R_3' \in \mathscr{C}(1)$, $R_3 \neq R_3'$, *such that the following conditions are true:*

- (a) $ON_2 \parallel T_{\mathscr{C}}(R_3)$ *and* $ON_1 \parallel T_{\mathscr{C}}(R_3')$, *i.e.,* $OR_3 \parallel T_{\mathscr{C}}(N_1')$, *by Claim 5.12;*
- (b) $R_3R_3' \nparallel \omega$ *and* $R_3R_3' \nparallel \mathbf{k}$, *i.e.,* R_3R_3' *gives the direction of the parallel projection onto* ω *and this direction must be non-degenerate;*
- (c) R_3, R_3', N_3 *are collinear, i.e.,* $\Pi_{\mathbf{v}}(R_3) = \Pi_{\mathbf{v}}(R_3') = N_3$.

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[13] Given $Q \in \mathscr{C}$, by (5.18) we know that $OP \parallel T_{\mathscr{C}}(Q) \Leftrightarrow x_P x_Q + y_P y_Q = 0$. So, having $N_1, N_2 \in \omega$ with $ON_1 \perp ON_2$, it follows that $ON_1 \parallel T_{\mathscr{C}}(N_2)$ and $ON_2 \parallel T_{\mathscr{C}}(N_1)$.

[14] In the following will not distinguish between these two possibilities because, by Rem. 5.16, we know that the triads R_1, R_2, R_3 and R_1', R_2', R_3' are equivalent. So will assume (6.4).

[15] Given $Q = (x_Q, y_Q, z_Q) \in \mathscr{C}$ and $M_1, M_2 \in \omega$ such that $OM_1 \nparallel OM_2$, we have that $OM_1, OM_2 \parallel T_{\mathscr{C}}(Q) \Leftrightarrow x_Q = y_Q = 0$. But the latter condition is equivalent to $OQ \perp \omega$.

6.1. **Explicit determination of $\Pi_{\mathbf{v}}$ in the circular case.** To proceed, we may suppose that the coordinate axes are oriented in ω such that

$$N_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad N_3 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \tag{6.9}$$

In this way we have

$$\overrightarrow{ON_3} = x\,\overrightarrow{ON_1} + y\,\overrightarrow{ON_2}. \tag{6.10}$$

Then, taking into account (5.18), we see that (a) in Cond. 6.2 is satisfied iff $R_3 \in \mathscr{C} \cap \{y = 0\}$ and $R_3' \in \mathscr{C} \cap \{x = 0\}$. Thus we can express R_3 and R_3' in the form

$$R_3 = \begin{pmatrix} \delta \\ 0 \\ \alpha \end{pmatrix} \quad \text{and} \quad R_3' = \begin{pmatrix} 0 \\ \delta' \\ \beta \end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}), \tag{6.11}$$

where

$$\delta, \delta' \in \{-1, 1\}. \tag{6.12}$$

Assuming (6.11) and (6.12), we certainly have $R_3 \neq R_3'$ and $R_3R_3' \nparallel \mathbf{k}$. This means that (b) of Cond. 6.2 holds iff

$$\alpha \neq \beta. \tag{6.13}$$

Besides, (c) of Cond. 6.2 is verified iff $N_3 = R_3 + t\,\overrightarrow{R_3R_3'}$ for some $t \in \mathbb{R}$, i.e.,

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \\ \alpha \end{pmatrix} + t \begin{pmatrix} -\delta \\ \delta' \\ \beta - \alpha \end{pmatrix} \quad \text{for some } t \in \mathbb{R}. \tag{6.14}$$

Claim 6.3. *Suppose* (6.14) *holds with* $\delta, \delta' = \pm 1$. *Then* x, y *satisfy*

$$(x + y + 1)(x + y - 1)(x - y + 1)(x - y - 1) = 0. \tag{6.15}$$

If we further assume that $\alpha, \beta, \beta - \alpha \neq 0$ *then* $x, y \neq 0$.

Proof. Let $\bar{t}$ be the solution of (6.14). The first two equations of (6.14) then give

$$x = \delta - \delta\bar{t} \quad \text{and} \quad y = \delta'\bar{t}. \tag{6.16}$$

Hence, for $\delta' \neq 0$, we find that

$$x + \frac{\delta}{\delta'}\,y - \delta = 0. \tag{6.17}$$

Assuming $\delta, \delta' = \pm 1$, it is clear that (6.15) holds because one of the factors cancels out.[16] Finally, assuming $\alpha, \beta, \beta - \alpha \neq 0$ the third equation of (6.14) implies $\bar{t} \neq 0, 1$. Then the first two equations of (6.14) give $x, y \neq 0$. □

Conversely, we have the following:

Claim 6.4. *Let us suppose* $x, y \neq 0$ *are such that*

$$(x + y + 1)(x + y - 1)(x - y + 1)(x - y - 1) = 0. \tag{6.18}$$

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[16] If $x, y \neq 0$ at most one of the factors of (6.15) can vanish.

Then, there exist $\delta, \delta' = \pm 1$ *and* $\alpha, \beta \neq 0$, $\beta - \alpha \neq 0$, *such that* (6.14) *holds. More precisely, the constants* δ *and* δ' *are uniquely determined by*

$$\delta = \frac{x^2 - y^2 + 1}{2x}, \quad \delta' = \frac{y^2 - x^2 + 1}{2y}; \tag{6.19}$$

α, β *and are determined, up to a common non-zero factor, by*

$$\alpha = \lambda, \quad \beta = \lambda \left(1 - \frac{\delta'}{y}\right) \quad \text{with } \lambda \neq 0 \text{ arbitrary.} \tag{6.20}$$

Proof. Since $x, y \neq 0$, only one of the factors of the left-hand side of (6.18) is zero. There are therefore unique $\delta, \delta' \in \{-1, 1\}$ such that

$$x + \frac{\delta}{\delta'} y - \delta = 0. \tag{6.21}$$

By setting $t = y/\delta'$, we have therefore $x = \delta - \delta t$ and $y = \delta' t$, i.e., the first two equations of (6.14). To verify the third it is necessary and sufficient that

$$\beta = \alpha \left(1 - \frac{1}{t}\right) = \alpha \left(1 - \frac{\delta'}{y}\right). \tag{6.22}$$

Noting that $\delta'/y \neq 0, 1$, from (6.22) we find that $\beta, \beta - \alpha \neq 0$ iff $\alpha \neq 0$. It is therefore clear that α, β must satisfy (6.20). Finally, to prove (6.19), noting (6.21) we can write

$$\delta' = \frac{y\delta}{\delta - x}. \tag{6.23}$$

Then, since $\delta^2 = \delta'^2 = 1$, we easily have

$$1 - 2\delta x + x^2 = y^2, \tag{6.24}$$

which allows us to obtain δ as in the first expression of (6.19). Similarly we get δ'. ☐

Summarizing up, we may conclude the following:

Claim 6.5. *Assume* (6.1) *is verified and that* $\overrightarrow{ON_3} = x\,\overrightarrow{ON_1} + y\,\overrightarrow{ON_2}$ *with* $x, y \neq 0$. *Then there exists a cylindrical Pohlke's projection* $\Pi_{\mathbf{v}}$ *for* ON_1, ON_2, ON_3 *if and only if*

$$g(x,y) \overset{\text{def}}{=} (x+y+1)(x+y-1)(x-y+1)(x-y-1) = 0, \tag{6.25}$$

and this projection is unique up to equivalence in the sense of Def. 4.4
More precisely, if (6.25) *holds, we have* $\mathscr{C} = \mathscr{C}(1)$ *and the projection direction may be parallel to any vector of the form*

$$\mathbf{v} = \overrightarrow{ON_1} + \eta \overrightarrow{ON_2} + \lambda\,\mathbf{k} \quad \text{with } \lambda \neq 0 \text{ arbitrary} \tag{6.26}$$

and $\eta = \operatorname{sgn}\left(xy(x^2 + y^2 - 1)\right)$. [4]

Proof. Suppose there exists a cylindrical Pohlke's projection $\Pi_{\mathbf{v}}$ for ON_1, ON_2 ON_3. From Claim 6.1, we know that $\rho = 1$, i.e., $\mathscr{C} = \mathscr{C}(1)$. Besides, from Cond. 6.2, it follows that (6.14) must be verified for appropriate values of the constants $\delta, \delta' \in \{-1, 1\}$ and $\beta \neq \alpha$. Then, applying Claim 6.3, with get the condition $g(x,y) = 0$.
Conversely, let us suppose $g(x,y) = 0$. Having assumed $x, y \neq 0$, by Claim 6.4 we deduce that (6.14) is verified for $\delta, \delta' \in \{-1, 1\}$ given by (6.19) and α, β as in (6.20). By

Cond. 6.2 there are then infinite cylindrical Pohlke's projections (all with $\mathscr{C} = \mathscr{C}(1)$, for Claim 6.1) and by (6.9), (6.11) and (6.20) the projections directions must be parallel to

$$\overrightarrow{R_3R_3'} = -\delta\,\overrightarrow{ON_1} + \delta'\,\overrightarrow{ON_2} - \lambda\frac{\delta'}{y}\,\mathbf{k}, \tag{6.27}$$

with the given $\delta, \delta' \in \{-1, 1\}$ and $\lambda \neq 0$ arbitrary. So these projections are all equivalent, in the sense of Def. 4.4. Finally, using (6.19) and condition (6.18), we can express $\delta\delta'$ as

$$\delta\delta' = \frac{1-(x^2-y^2)^2}{4xy} = \frac{1-x^2-y^2}{2xy}. \tag{6.28}$$

We therefore deduce that

$$-\delta\delta' = \begin{cases} 1, & \text{if} \quad xy(x^2+y^2-1) > 0 \\ -1, & \text{if} \quad xy(x^2+y^2-1) < 0 \end{cases} \tag{6.29}$$

from which we immediately obtain the simplified form (6.26). □

7. PROOF OF THEOREM 4.9

(1) ⇒ (2). It is sufficient to apply Claim 5.15.
Indeed, we are assuming $OP_i \nparallel OP_j$ $(1 \le i < j \le 3)$. Hence, taking into account the conditions (4.6), (4.7) of Def. 4.5, from the first part of Claim 5.15 we get:

$$\Pi_{\mathbf{v}}(Q_1) = P_1,\ \Pi_{\mathbf{v}}(Q_2) = P_2 \quad \text{and} \quad OQ_1 \parallel T_{\mathscr{C}}(Q_2) \ \Rightarrow\ \Pi_{\mathbf{v}}(\mathcal{C}(Q_1, Q_2)) = \mathcal{E}_{P_1,P_2}, \tag{7.1}$$

$$\Pi_{\mathbf{v}}(Q_2) = P_2,\ \Pi_{\mathbf{v}}(Q_3) = P_3 \quad \text{and} \quad OQ_2 \parallel T_{\mathscr{C}}(Q_3) \ \Rightarrow\ \Pi_{\mathbf{v}}(\mathcal{C}(Q_2, Q_3)) = \mathcal{E}_{P_2,P_3}. \tag{7.2}$$

Noting that $\Pi_{\mathbf{v}}(Q_1') = P_1$, we also find that

$$\Pi_{\mathbf{v}}(Q_3) = P_3,\ \Pi_{\mathbf{v}}(Q_1') = P_1 \quad \text{and} \quad OQ_3 \parallel T_{\mathscr{C}}(Q_1') \ \Rightarrow\ \Pi_{\mathbf{v}}(\mathcal{C}(Q_3, Q_1')) = \mathcal{E}_{P_3,P_1}. \tag{7.3}$$

Furthermore, since $\Pi_{\mathbf{v}}$ is non-degenerate, from the second part of Claim 5.15 we deduce that $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ are tangent to $\mathscr{T}_{\mathbf{v}} = \Pi_{\mathbf{v}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{v}})$. In conclusion

$$\mathcal{T} = \mathscr{T}_{\mathbf{v}}$$

is a cylindrical Pohlke's conic for OP_1, OP_2, OP_3.

(2) ⇒ (1). This implication can be obtained by applying part 1) of Claim 5.10 and then Claim A.1 of the Appendix. Indeed, let $\mathcal{T} = \mathcal{T}_- \cup \mathcal{T}_+$ be a cylindrical Pohlke's conic for OP_1, OP_2, OP_3. To begin with, we fix

$$\rho = d/2, \quad \text{with} \quad d \overset{\text{def}}{=} \text{distance between } \mathcal{T}_- \text{ and } \mathcal{T}_+, \tag{7.4}$$

and then a non-zero vector $\mathbf{w} \parallel \omega$ such that $\mathcal{T}_-, \mathcal{T}_+ \parallel \mathbf{w}$. Next we set

$$\mathbf{v} = \mathbf{w} + \lambda\,\mathbf{k}, \tag{7.5}$$

with $\lambda \neq 0$ arbitrary. This means that

$$\mathcal{T}_- \cup \mathcal{T}_+ = \Pi_{\mathbf{v}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{v}}) = \mathscr{T}_{\mathbf{v}}. \tag{7.6}$$

After that, we consider the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$, which are tangent to $\mathscr{T}_{\mathbf{v}}$.
Starting with $\mathcal{E}_{P_1,P_2}$, by part 1) of Claim 5.10 there is a plane π, through the origin O, such that $\mathscr{C} \cap \pi$ is an ellipse and $\Pi_{\mathbf{v}}(\mathscr{C} \cap \pi) = \mathcal{E}_{P_1,P_2}$. Furthermore, there are $Q_1, Q_2 \in \mathscr{C} \cap \pi$ such that $\Pi_{\mathbf{v}}(Q_1) = P_1$, $\Pi_{\mathbf{v}}(Q_2) = P_2$ and OQ_1, OQ_2 are conjugate semi-diameters of the

ellipse $\mathscr{C} \cap \pi$. With the notation of Def. 5.13, this later fact implies $OQ_1 \parallel T_{\mathcal{C}(Q_1,Q_2)}(Q_2)$. Then

$$OQ_1 \parallel T_{\mathcal{C}(Q_1,Q_2)}(Q_2) \quad \text{and} \quad T_{\mathcal{C}(Q_1,Q_2)}(Q_2) \subset T_{\mathscr{C}}(Q_2) \ \Rightarrow\ OQ_1 \parallel T_{\mathscr{C}}(Q_2). \tag{7.7}$$

So the first condition of (4.7) is satisfied. To proceed further, we consider $\mathcal{E}_{P_2,P_3}$. Again from part 1) of Claim 5.10 we can find a plane $\tilde{\pi}$, through O and Q_2, such that $\mathscr{C} \cap \tilde{\pi}$ is an ellipse and $\Pi_{\mathbf{v}}(\mathscr{C} \cap \tilde{\pi}) = \mathcal{E}_{P_2,P_3}$. Besides, we can also find a point $Q_3 \in \mathscr{C} \cap \tilde{\pi}$ such that $\Pi_{\mathbf{v}}(Q_3) = P_3$ and OQ_2, OQ_3 are conjugate semi-diameters of $\mathscr{C} \cap \tilde{\pi}$. As above, we deduce that

$$OQ_2 \parallel T_{\mathscr{C}}(Q_3). \tag{7.8}$$

So the second condition of (4.7) holds.

Finally, we consider the ellipse $\mathcal{E}_{P_3,P_1}$. Noting that

$$\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathscr{C} = \{Q_1, Q_1'\} \quad \text{with } Q_1, Q_1' \ \pi_{\mathbf{v}} - \text{symmetric}, \tag{7.9}$$

and reasoning as above, it is clear that at least one of the following must be true:

$$OQ_3 \parallel T_{\mathscr{C}}(Q_1) \quad \text{or} \quad OQ_3 \parallel T_{\mathscr{C}}(Q_1'). \tag{7.10}$$

But, by Claim A.1, we cannot have the sequence

$$OQ_1 \parallel T_{\mathscr{C}}(Q_2),\ OQ_2 \parallel T_{\mathscr{C}}(Q_3) \ \text{and}\ OQ_3 \parallel T_{\mathscr{C}}(Q_1), \tag{7.11}$$

with $Q_1, Q_2, Q_3 \in \mathscr{C}$. Hence the second (and only the second) of (7.10) is true.

In conclusion, we have found three points $Q_1, Q_2, Q_3 \in \mathscr{C} = \mathscr{C}(\rho)$ such that (4.6) and (4.7) hold. This means that the projection $\Pi_{\mathbf{v}}$, thus determined, is a cylindrical Pohlke's projection for OP_1, OP_2, OP_3.

7.1. **The equivalence of (1),(2) with (3).** To prove that (1),(2) $\Leftrightarrow$ (3), we use the equivalence **(1)** $\Leftrightarrow$ **(2)** just proven and resort to a suitable circular case. More precisely, let $N_1, N_2 \in \omega$ such that

$$ON_1 \perp ON_2 \quad \text{and} \quad |ON_1| = |ON_2| = 1. \tag{7.12}$$

Since $OP_1 \nparallel OP_2$, we may consider the affine transformation $\Phi : \omega \to \omega$ defined by

$$\Phi(O + x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}\,) \overset{\text{def}}{=} O + x\,\overrightarrow{ON_1} + y\,\overrightarrow{ON_2} \quad \text{for} \quad x, y \in \mathbb{R}. \tag{7.13}$$

It is clear that $\Phi(P_1) = N_1$, $\Phi(P_2) = N_2$. Besides, if $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$, then

$$N_3 \overset{\text{def}}{=} \Phi(P_3) = O + h\,\overrightarrow{ON_1} + k\,\overrightarrow{ON_2}. \tag{7.14}$$

Hence, having $OP_3 \nparallel OP_1, OP_2$, we find

$$\overrightarrow{ON_3} = h\,\overrightarrow{ON_1} + k\,\overrightarrow{ON_2} \quad \text{and} \quad ON_3 \nparallel ON_1, ON_2 \ \text{(i.e., } h, k \neq 0). \tag{7.15}$$

As it is known, an affine transformation maps conjugate semi-diameters of a central conic into conjugate semi-diameters of the transformed conic. This means that $\Phi(\mathcal{E}_{P_1,P_2}) = \mathcal{E}_{N_1,N_2}$, $\Phi(\mathcal{E}_{P_2,P_3}) = \mathcal{E}_{N_2,N_3}$ and $\Phi(\mathcal{E}_{P_3,P_1}) = \mathcal{E}_{N_3,N_1}$. Besides, if $\mathcal{T} = \mathcal{T}_- \cup \mathcal{T}_+$ is a cylindrical Pohlke's conic for OP_1, OP_2, OP_3 (that is, $\mathcal{T}_-, \mathcal{T}_+$ are distinct and parallel lines, tangent to $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$) then $\Phi(\mathcal{T}) = \Phi(\mathcal{T}_-) \cup \Phi(\mathcal{T}_+)$ is cylindrical Pohlke's conic for ON_1, ON_2, ON_3 (that is, $\Phi(\mathcal{T}_-), \Phi(\mathcal{T}_+)$ are distinct and parallel lines, tangent to $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$, $\mathcal{E}_{N_3,N_1}$). Finally, the converse is also true, because $\Phi^{-1} : \omega \to \omega$ is still an affine transformation. Hence, according to Def. 4.8, we can state the following:

Claim 7.1. *Let* $\Phi : \omega \to \omega$ *be the affine transformation defined in* (7.13).
If $\mathcal{T}$ *is a cylindrical Pohlke's conic for* OP_1, OP_2, OP_3, *then* $\Phi(\mathcal{T})$ *is a cylindrical Pohlke's conic for* ON_1, ON_2, ON_3 *and vice versa.*

(1),(2) $\Rightarrow$ **(3)**. Now let us suppose that (2) holds, namely that there is a cylindrical Pohlke's conic $\mathcal{T}$ for OP_1, OP_2, OP_3. Then

$$\mathcal{T}_o = \Phi(\mathcal{T}) \tag{7.16}$$

is a cylindrical Pohlke's conic for ON_1, ON_2, ON_3. Having already proved that **(2)** $\Rightarrow$ **(1)**, there is then a cylindrical Pohlke's projection for ON_1, ON_2, ON_3. By (7.12) and (7.15) we can apply Claim 6.5 to ON_1, ON_2, ON_3. We therefore conclude that h, k must satisfy (4.8).

(3) $\Rightarrow$ **(1),(2)**. Conversely, let us suppose that (3) holds, i.e., $h, k \neq 0$ satisfy the condition (4.8). Then, by Claim 6.5, there is a cylindrical Pohlke's projection for ON_1, ON_2, ON_2 . By the implication **(1)** $\Rightarrow$ **(2)**, we deduce the existence of a cylindrical Pohlke's conic, say $\mathcal{T}_o$, for ON_1, ON_2, ON_3. Then

$$\mathcal{T} = \Phi^{-1}(\mathcal{T}_o) \tag{7.17}$$

is a cylindrical Pohlke's conic for OP_1, OP_2, OP_3. We have thus shown that (2) holds.

7.2. **Uniqueness of** $\Pi_{\mathbf{v}}, \mathcal{T}$ **and proof of** (2.3), (2.4)**.** By **(1)** $\Rightarrow$ **(2)**, we already know that if $\Pi_{\mathbf{v}}$ is a cylindrical Pohlke's projection for OP_1, OP_2, OP_3, then $\mathscr{T}_{\mathbf{v}} = \Pi_{\mathbf{v}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{v}})$ is cylindrical Pohlke's conic for OP_1, OP_2, OP_3. In the circular case, i.e., for non-parallel ON_1, ON_2, ON_3 such that (6.1) holds, we have the following:

Corollary 7.2. *With the assumptions of Claim 6.5, if* (6.25) *holds then the exists a unique cylindrical Pohlke's conic* $\mathcal{T}$ *for* ON_1, ON_2, ON_3 *and it is given by*

$$\mathcal{T} = \Pi_{\mathbf{v}}(\mathscr{C}(1) \cap \pi_{\mathbf{v}}) = \mathscr{T}_{\mathbf{v}}^{-} \cup \mathscr{T}_{\mathbf{v}}^{+}, \tag{7.18}$$

with $\mathbf{v}$ *as in* (6.26). *More precisely,* $\mathscr{T}_{\mathbf{v}}^{-}$ *and* $\mathscr{T}_{\mathbf{v}}^{+}$ *are the lines passing through the points*

$$O - \frac{\overrightarrow{ON_1} - \eta\,\overrightarrow{ON_2}}{\sqrt{2}} \quad \text{and} \quad O + \frac{\overrightarrow{ON_1} - \eta\,\overrightarrow{ON_2}}{\sqrt{2}}, \tag{7.19}$$

respectively, and parallel to $\overrightarrow{ON_1} + \eta\,\overrightarrow{ON_2}$.

Proof. For Claim 6.1 in the circular case we have $\mathscr{C} = \mathscr{C}(1)$. From implication **(1)** $\Rightarrow$ **(2)** and Claim 6.5, we have then that $\mathcal{T} = \Pi_{\mathbf{v}}(\mathscr{C}(1) \cap \pi_{\mathbf{v}})$ (with $\mathbf{v}$ as in (6.26)) gives a cylindrical Pohlke's conic for ON_1, ON_2, ON_3. By (6.26) it is also clear that $\mathcal{T}$ thus determined does not depend on the choice of $\lambda \neq 0$[17] and taking into account Def. 5.5 we have $\mathcal{T} = \mathscr{T}_{\mathbf{v}}^{-} \cup \mathscr{T}_{\mathbf{v}}^{+}$, with $\mathscr{T}_{\mathbf{v}}^{-}, \mathscr{T}_{\mathbf{v}}^{+} \parallel \overrightarrow{ON_1} + \eta\,\overrightarrow{ON_2}$. Moreover, since $\eta = \pm 1$ and we are assuming $|ON_1| = |ON_2| = 1$, with $ON_1 \perp ON_2$, it follows that $\|\overrightarrow{ON_1} - \eta\,\overrightarrow{ON_2}\| = \sqrt{2}$. We can therefore easily see (as in formula (5.10)) that the lines $\mathscr{T}_{\mathbf{v}}^{-}$, $\mathscr{T}_{\mathbf{v}}^{+}$ pass through the points given by (7.19).

It remains to be shown that the cylindrical Pohlke's conic is unique. For this purpose, we can use the same arguments of the implication **(2)** $\Rightarrow$ **(1)** proved above. In fact, given a cylindrical Pohlke's conic $\mathcal{F}$ (for N_1, N_2, N_3), we can prove that there exists a cylindrical Pohlke's projection $\Pi_{\mathbf{u}}$ (for N_1, N_2, N_3) and that, as in (7.6),

$$\mathcal{F} = \mathcal{F}_{-} \cup \mathcal{F}_{+} = \Pi_{\mathbf{u}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{u}}), \tag{7.20}$$

---

[17] This follows also from Claim 5.7.

with ρ given as in (7.4). Now we can observe that (7.20) requires $\rho = 1$, because $\mathcal{E}_{N_1,N_2}$ is a circle with unit radius and $\mathcal{F}$ is tangent to $\mathcal{E}_{N_1,N_2}$. Besides, since we are in the circular case, by Claim 6.5 the projection $\Pi_{\mathbf{u}}$ is uniquely determined up to equivalence in the sense of Def. 4.4. In other words, $\Pi_{\mathbf{u}}$ is equivalent to any projection $\Pi_{\mathbf{v}}$ determined by (6.26). Then, by Claim 5.7, the right hand side of (7.20) is independent of $\mathcal{F}$, i.e., $\mathcal{F} = \mathcal{T}$. We have thus demonstrated that in the circular case $\mathcal{T}$ is unique. □

We can now prove the uniqueness of $\Pi_{\mathbf{v}}$ and $\mathcal{T}$ in Theorem 4.9.

Uniqueness of $\mathcal{T}$. The uniqueness of cylindrical Pohlke's conic $\mathcal{T}$ for OP_1, OP_2, OP_3 follows immediately from the uniqueness in the circular case just proved in Cor. 7.2. In fact, applying Claim 7.1, we know that $\mathcal{T}$ is a cylindrical Pohlke's conic for OP_1, OP_2, OP_3 if and only if $\Phi(\mathcal{T})$ is a cylindrical Pohlke's conic for ON_1, ON_2, ON_3.

Uniqueness of $\Pi_{\mathbf{v}}$. So far we have shown that there is a unique cylindrical Pohlke's conic $\mathcal{T} = \mathcal{T}_- \cup \mathcal{T}_+$ for OP_1, OP_2, OP_3. Now, let $\Pi_{\mathbf{v}}$ be a cylindrical Pohlke's projection (according to Def. 4.5) for OP_1, OP_2, OP_3. In **(1)** ⇒ **(2)** we have shown that $\Pi_{\mathbf{v}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{v}})$ gives a cylindrical Pohlke's conic for OP_1, OP_2, OP_3. Then, we must have

$$\Pi_{\mathbf{v}}(\mathscr{C}(\rho) \cap \pi_{\mathbf{v}}) = \mathcal{T}. \tag{7.21}$$

Having proved (7.21) it is then sufficient to observe, as in **(2)** ⇒ **(1)**, that ρ is uniquely determined by $\mathcal{T}$ (see (7.4)) and we must also have

$$\mathbf{v} \parallel \mathbf{w} + \lambda \mathbf{k} \quad \text{for some} \quad \lambda \neq 0, \tag{7.22}$$

where $\mathbf{w}$ is a non-zero vector such that $\mathcal{T}_-, \mathcal{T}_+ \parallel \mathbf{w}$. This means that the cylindrical Pohlke's projection for OP_1, OP_2, OP_3 is uniquely determined up to equivalence in the sense of Def. 4.4.

Proof of (2.3), (2.4). Again for Claim 7.1, both these formulas follows immediately, via the inverse of the affine transformation Φ defined in (7.13), from the analogous formulas demonstrated in the circular case. See Claim 6.5 and Cor. 7.2.

A. APPENDIX

In Def. 4.5 we require the condition

$$OQ_1 \parallel T_{\mathscr{C}}(Q_2),\ OQ_2 \parallel T_{\mathscr{C}}(Q_3) \ \text{ and } \ OQ_3 \parallel T_{\mathscr{C}}(Q_1'), \tag{A.1}$$

where $Q_1' \in \mathscr{C}(\rho)$ is the point $\pi_{\mathbf{v}}$ – *symmetric* of Q_1. It is easy to prove that in the last term of (A.1) we cannot replace Q_1' with Q_1. In fact, we have:

Claim A.1. *There does not exist* $Q_1, Q_2, Q_3 \in \mathscr{C}(\rho)$ *such that*

$$OQ_1 \parallel T_{\mathscr{C}}(Q_2),\ OQ_2 \parallel T_{\mathscr{C}}(Q_3) \ \text{ and } \ OQ_3 \parallel T_{\mathscr{C}}(Q_1). \tag{A.2}$$

Proof. Writing $Q_1 = (x_1, y_1, z_1)$, $Q_2 = (x_2, y_2, z_2)$ and $Q_3 = (x_3, y_3, z_3)$, by (5.18) we can reformulate (A.2) in the equivalent form:

$$\begin{cases} x_1x_2 + y_1y_2 = 0 \\ x_2x_3 + y_2y_3 = 0 \\ x_1x_3 + y_1y_3 = 0 \end{cases} \tag{A.3}$$

Then, assuming $Q_1, Q_2 \in \mathscr{C}(\rho)$ are such that $OQ_1 \parallel T_{\mathscr{C}}(Q_2)$ (i.e., the first equation of (A.3) holds), we easily see that there does not exist $Q_3 \in \mathscr{C}(\rho)$ such that $OQ_2 \parallel T_{\mathscr{C}}(Q_3)$ and $OQ_3 \parallel T_{\mathscr{C}}(Q_1)$ (i.e., the last two equations of (A.3) hold). In fact, since $Q_1, Q_2 \in \mathscr{C}(\rho)$, the first equation of (A.3) gives

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \pm\rho^2 \neq 0. \tag{A.4}$$

Then the last two of (A.3) imply $x_3 = y_3 = 0$. So $Q_3 \notin \mathscr{C}(\rho)$, regardless of $\rho > 0$. ☐

In particular, Claim A.1 has the following consequence:

Corollary A.2. *If* $Q_1, Q_2, Q_3 \in \mathscr{C}$ *satisfy* (A.1), *then* $Q_i \neq Q_i'$ *for* $1 \leq i \leq 3$.

Proof. In fact, if $Q_i = Q_i'$ for some $1 \leq i \leq 3$, renaming the points Q_1, Q_2, Q_3 we get (A.2). For instance, let us suppose $Q_2 = Q_2'$. Noting that $Q_2 = Q_2'$ and $OQ_2 \parallel T_{\mathscr{C}}(Q_3) \Rightarrow OQ_2 \parallel T_{\mathscr{C}}(Q_3')$ and that $OQ_3 \parallel T_{\mathscr{C}}(Q_1') \Leftrightarrow OQ_3' \parallel T_{\mathscr{C}}(Q_1)$, we merely set

$$R_1 = Q_1,\ R_2 = Q_2,\ R_3 = Q_3'.$$

Then $R_1, R_2, R_3 \in \mathscr{C}(\rho)$ satisfy $OR_1 \parallel T_{\mathscr{C}}(R_2),\ OR_2 \parallel T_{\mathscr{C}}(R_3)$ and $OR_3 \parallel T_{\mathscr{C}}(R_1)$. ☐

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