A short proof of Pohlke-Schwarz's theorem via Pohlke's theorem

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Abstract

We give a proof of Pohlke-Schwarz's theorem of oblique axonometry with explicit formulae for the reference tetrahedron and the direction of projection onto the image plane.

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1 Introduction

In 1864 H.A. Schwarz [10] published the proof the following generalized statement of Pohlke's fundamental theorem of oblique axonometry [9]:

three arbitrary straight line segments OP_1, OP_2, OP_3 in a plane, originating from a point O and which are not contained in a line, can be considered as the parallel projection of three edges OQ_1, OQ_2, OQ_3 of a tetrahedron that is similar to a given tetrahedron,

Several purely geometric proofs and, in a few instances, analytic proofs were given. See, among the others, [1], [2], [3], [5] [8]. Here, applying the results of [7], we give a straightforward proof of the above statement together with explicit formulae for the edges OQ_1, OQ_2, OQ_3 of reference tetrahedron and the direction of the parallel projection onto the image plane.

As we did in [7] for Pohlke's theorem, we reformulate the Pohlke-Schwarz's theorem as a result of linear algebra for square matrices of order 3. For these reason, and to avoid repetitions, all the matrices that we consider from now on are 3×3, real matrices. If A is such a matrix, with A^i and A_i $(1 \leq i \leq 3)$ we denote, respectively, columns and rows vectors of A.

1.1 Reformulation of the problem

To begin with, we introduce a cartesian system of coordinate axes x, y, z such that

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad P_i = \begin{pmatrix} \mathbf{x}_i \\ \mathbf{y}_i \\ 0 \end{pmatrix} \quad (1 \leq i \leq 3) \tag{1.1}$$

are points of the image plane $\{z = 0\}$ and then we define the matrix

$$\mathbf{A} \overset{\text{def}}{=} \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

Furthermore, we represent a given tetrahedron, with a vertex at the origin O, by a 3×3 matrix whose columns are the coordinates of the other vertices.

After this, if the matrices S, T represent two given tetrahedrons, we say that:

Definition 1.1 *S, T are "geometrically similar" if $T = HS$ with H a nonzero multiple of an orthogonal matrix, i.e., $HH^t = \mu I$ for some $\mu > 0$.*

Remark 1.2 This is clearly an equivalence relation, but it is different from the usual definition of "similar" matrices. See [6, Definition 5.1.1]. □

Pohlke-Schwarz's theorem can now be stated as follows:

Theorem 1.3 *Assume that* $\operatorname{rank} \mathbf{A} = 2$, *and let S be an invertible matrix. Then, there exist a matrix $\mathbf{B}$, geometrically similar to S,*

$$\mathbf{B} = \begin{pmatrix} \mathbf{x}_1' & \mathbf{x}_2' & \mathbf{x}_3' \\ \mathbf{y}_1' & \mathbf{y}_2' & \mathbf{y}_3' \\ \mathbf{z}_1' & \mathbf{z}_2' & \mathbf{z}_3' \end{pmatrix}, \tag{1.3}$$

and a parallel projection $\mathbf{\Pi}$ onto the plane $\{z = 0\}$ such that $\mathbf{\Pi}(\mathbf{B}^i) = \mathbf{A}^i$, i.e.,

$$\mathbf{\Pi} \begin{pmatrix} \mathbf{x}_i' \\ \mathbf{y}_i' \\ \mathbf{z}_i' \end{pmatrix} = \begin{pmatrix} \mathbf{x}_i \\ \mathbf{y}_i \\ 0 \end{pmatrix}, \quad 1 \le i \le 3. \tag{1.4}$$

2 Proof of Theorem 1.3

Since S is invertible, we can define the matrix

$$A \overset{\text{def}}{=} \mathbf{A} S^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Noting that $\operatorname{rank} A = 2$ and $A_3 = (0, 0, 0)$, we can apply to A the Pohlke's theorem in the form stated in [7, Theorem 1.1]. This means that:

there exist a matrix B, with orthogonal columns of equal norm, and a parallel projection Π onto the image plane $\{z = 0\}$ such that

$$\Pi(B^i) = A^i, \quad 1 \le i \le 3. \tag{2.2}$$

In particular, we have

$$\operatorname{rank}(B - A) = 1. \tag{2.3}$$

Then, setting

$$\mathbf{B} = BS, \tag{2.4}$$

we find that

$$\operatorname{rank}(\mathbf{B} - \mathbf{A}) = \operatorname{rank}(B - A)S = 1, \tag{2.5}$$

because S is invertible.

It follows that there exist a column vector $\mathbf{U}$ and real coefficients ν_i such that

$$\mathbf{B}^i - \mathbf{A}^i \,=\, \nu_i \,\mathbf{U}\,, \quad 1 \leq i \leq 3\,. \tag{2.6}$$

Moreover, $\mathbf{U} \notin \operatorname{span}\{\, \mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3\} = \{z = 0\}$ because $\operatorname{rank}\mathbf{B} = 3$.

Thus, we can define the parallel projection $\mathbf{\Pi}$, onto the image plane $\{z = 0\}$, in the direction of the column vector $\mathbf{U}$. Clearly, $\mathbf{\Pi}$ verifies

$$\mathbf{\Pi}(\mathbf{B}^i) = \mathbf{A}^i, \quad 1 \leq i \leq 3\,. \tag{2.7}$$

Moreover, since B is a nonzero a multiple of an orthogonal matrix, $\mathbf{B} = BS$ is geometrically similar to S. This concludes the proof Theorem 1.3. □

Remark 2.1 *In the previous proof we have*

$$\mathbf{\Pi} \,\equiv\, \Pi\,. \tag{2.8}$$

In fact, $\mathbf{\Pi}$ and Π are both parallel projections onto the image plane $\{z = 0\}$.

Besides, by (2.3), there exist a column vector U and real coefficients μ_i such that

$$B^i - A^i = \mu_i U\,, \quad 1 \leq i \leq 3\,. \tag{2.9}$$

Introducing the row vector $\mu = (\mu_1, \mu_2, \mu_3)$, we have $B - A = U\mu$. Hence, we deduce that

$$\mathbf{B} - \mathbf{A} = (B - A)S = U\mu S = \big(\, (\mu S^1)U,\, (\mu S^2)U,\, (\mu S^3)U\,\big)\,, \tag{2.10}$$

i.e., $\mathbf{B}^i - \mathbf{A}^i = (\mu S^i)U$. Thus $\mathbf{\Pi}$, Π project in the same direction onto the image plane $\{z = 0\}$ and formula (2.6) can be verified by taking $\mathbf{U} = U$. □

As by product of Remark 2.1, we can state a simple generalization, for oblique system of coordinate-axes, of the Gauss' fundamental theorem of orthogonal axonometry (see [4] for more general results in this direction). More precisely, denoting with $\Pi_\perp$ the orthogonal projection onto the image plane $\{z = 0\}$, we have:

Corollary 2.2 *Let S be invertible and let $\mathbf{A}$, A be the matrices defined in* (1.2), (2.1)*. Then, there exists a matrix $\mathbf{B}$, geometrically similar to S, such that $\Pi_\perp(\mathbf{B}^i) = \mathbf{A}^i$ if and only if*

$$\|A_1\| = \|A_2\| \neq 0 \quad \textit{with} \quad A_1 \perp A_2\,. \tag{2.11}$$

Proof: Taking into account Remark 2.1, it is enough to apply [7, Proposition 1.2] to the rows of the matrix $A = \mathbf{A}S^{-1}$. □

2.1 Reference tetrahedron and direction of projection

Following the steps of the proof of Theorem 1.3, we can now determine the matrix $\mathbf{B}$ and a column vector $\mathbf{U}$ representing the direction of the parallel projection $\mathbf{\Pi}$ onto the image plane $\{z = 0\}$. We begin by setting:

$$A_1 = \mathbf{A}_1 S^{-1}\,, \quad A_2 = \mathbf{A}_2 S^{-1}. \tag{2.12}$$

That is $A_1 = (x_1, x_2, x_3)$, $A_2 = (y_1, y_2, y_3)$ as in (2.1).

Besides, as in formulae (3.6), (3.10), (3.21), (3.22) of [7], we define the quantities:

$$\gamma = \arccos\left(\frac{A_1 \cdot A_2}{\|A_1\| \, \|A_2\|}\right), \quad \lambda = \frac{\|A_1\|}{\|A_2\|}, \tag{2.13}$$

$$\eta = \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma}, \tag{2.14}$$

$$\nu = \pm\varrho \quad \text{with} \quad \varrho = \frac{\|A_1\|}{\lambda\sqrt{\eta}} = \frac{\|A_2\|}{\sqrt{\eta}}, \tag{2.15}$$

and, finally,

$$(\alpha, \beta) = \pm\left(\sqrt{\eta\,\lambda^2 - 1}\ , \ \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\ \right) \tag{2.16}$$

with the “signum” function:

$$\operatorname{sgn}(t) \doteqdot \begin{cases} 1 & \text{if} \quad t \geq 0\,, \\ -1 & \text{if} \quad t < 0\,. \end{cases} \tag{2.17}$$

Then, by the results of [7, Chapter 4], the matrix B and the direction U of the projection Π satisfying (2.2) are given by the relations:

$$B = \frac{1}{1+\alpha^2+\beta^2}\begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3-y_2x_3}{\nu} & \frac{y_1x_3-x_1y_3}{\nu} & \frac{x_1y_2-y_1x_2}{\nu} \end{pmatrix}, \tag{2.18}$$

$$U = \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix}. \tag{2.19}$$

Taking into account (2.4) and Remark 2.1, we may conclude by setting:

$$\mathbf{B} = BS\,, \quad \mathbf{U} = U. \tag{2.20}$$

In particular, we have

$$\mathbf{\Pi}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+\alpha z \\ y+\beta z \\ 0 \end{pmatrix}. \tag{2.21}$$

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