On Pohlke's type projections in the hyperbolic case

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(extended from the 11/21/22 version)

Abstract

Let OP_1, OP_2, OP_3 be three non-parallel segments in a plane ω. We find the necessary and sufficient conditions for the existence of the common inscribed ellipse $\mathcal{E}_{\mathtt{I}}$ and the common circumscribing hyperbola $\mathcal{H}_{\mathtt{C}}$ (both with center O) of the three ellipses having as conjugate semi-diameters the pairs $(OP_1, OP_2), (OP_2, OP_3)$ and (OP_3, OP_1). To this end we introduce a new parallel projection (hyperbolic Pohlke's projection) such that $\mathcal{E}_{\mathtt{I}}$ and $\mathcal{H}_{\mathtt{C}}$ are obtained as the outline of the image into ω of a suitable hyperboloid.

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1 Introduction

Given three non-parallel segments OP_1, OP_2, OP_3 originating at O and lying in a plane ω, we consider the three concentric ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ determined by the pairs of conjugate semi-diameter (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1), respectively.

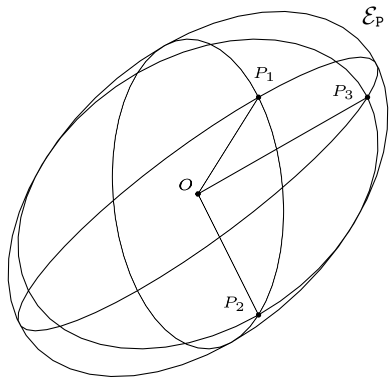
It is a simple consequence of Pohlke's fundamental theorem ([2], [5], [6], [9]) that there is *always* an ellipse with center O, here indicated with $\mathcal{E}_{\mathtt{P}}$ (Pohlke's ellipse), which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$ ([3], [4], [7]). [1] It is possible to show (see [8, Thm. 3.8]) that if

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}, \tag{1.1}$$

then a pair of conjugate semi-diameters of $\mathcal{E}_{\mathtt{P}}$ is given by the vectors

$$\frac{k\overrightarrow{OP_1} - h\overrightarrow{OP_2}}{\sqrt{h^2+k^2}} \quad \text{and} \quad \sqrt{\frac{1+h^2+k^2}{h^2+k^2}}\,\overrightarrow{OP_3}. \tag{1.2}$$

[1] If OQ_1, OQ_2, OQ_3 are congruent, mutually orthogonal segments and $\Pi : \mathbb{R}^3 \to \omega$ is a parallel projection such that $\Pi(Q_i) = P_i$ for $i = 1, 2, 3$, then the ellipse $\mathcal{E}_{\mathtt{P}}$ is the contour of $\Pi(S)$, where $S \subset \mathbb{R}^3$ is the sphere with center O containing Q_1, Q_2, Q_3. The existence and uniqueness of $\mathcal{E}_{\mathtt{P}}$ follows from Pohlke's theorem.



Pohlke's ellipse with $P_1 = (1, \frac{8}{5}), P_2(1, -2), \overrightarrow{OP_3} = 2\overrightarrow{OP_1} + \frac{4}{5}\overrightarrow{OP_2}$.

Again supposing that OP_1, OP_2, OP_3 are non-parallel, if we further assume that

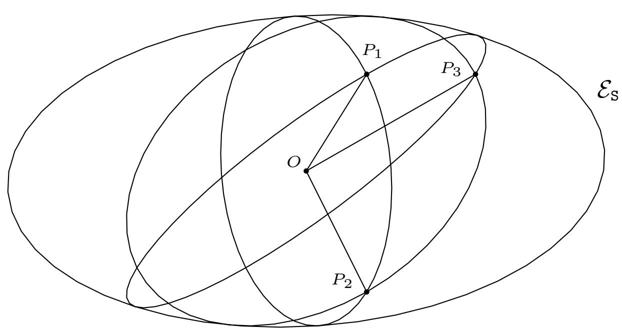
$$g(h,k) \stackrel{\text{def}}{=} (h+k+1)(h+k-1)(h-k+1)(h-k-1) > 0, \tag{1.3}$$

then there is a second concentric ellipse, other than $\mathcal{E}_{\mathtt{P}}$, which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$. We call this new ellipse the *secondary Pohlke's ellipse* $\mathcal{E}_{\mathtt{S}}$. It turns out that $\mathcal{E}_{\mathtt{S}}$ is unique and that (1.3) is also a necessary condition. See [7], [8] and [11]. In this case (see [8, Lem. 4.2]) a pair of conjugate semi-diameters of $\mathcal{E}_{\mathtt{S}}$ is given by the vectors

$$\frac{K\overrightarrow{OP_1} - H\overrightarrow{OP_2}}{\sqrt{H^2+K^2}} \quad \text{and} \quad \sqrt{\frac{g+H^2+K^2}{g(H^2+K^2)}}\,\Big(H\overrightarrow{OP_1} + K\overrightarrow{OP_2}\Big), \tag{1.4}$$

where

$$H(h,k) \stackrel{\text{def}}{=} h(h^2-k^2-1), \quad K(h,k) \stackrel{\text{def}}{=} k(h^2-k^2+1). \tag{1.5}$$



Secondary Pohlke's ellipse with $P_1 = (1, \frac{8}{5}), P_2(1, -2), \overrightarrow{OP_3} = 2\overrightarrow{OP_1} + \frac{4}{5}\overrightarrow{OP_2}$.

It is worth noting that in both the above cases the circumscribing ellipse ($\mathcal{E}_{\mathtt{P}}$ or $\mathcal{E}_{\mathtt{S}}$) is obtained as the contour of the parallel projection, into the drawing plane ω, of a suitable sphere with center O. In the present paper we investigated what happens when (1.3) does not hold, i.e., when the secondary Pohlke's ellipse $\mathcal{E}_{\mathtt{S}}$ does not exist. For this purpose we use the parallel projection, on the plane ω, of a suitable hyperboloid. We show that if

$$g(h,k) < 0 \tag{1.6}$$

and

$$f(h,k) \stackrel{\mathrm{def}}{=} (h^2+k^2-1)\big[(h^2-k^2)^2-1\big] \neq 0, \tag{1.7}$$

then there is a third conic, with center O, tangent to $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$.

Further, it turns out that this conic is an ellipse, say $\mathcal{E}_{\mathtt{I}}$, inscribed in $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ if h,k are such that $f(h,k)<0$. Conversely, it is a hyperbola, say $\mathcal{H}_{\mathtt{C}}$, which circumscribes the three ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ if $f(h,k)>0$. [2] Both $\mathcal{E}_{\mathtt{I}}$ and $\mathcal{H}_{\mathtt{C}}$, when they exit, are unique and the conditions (1.6), (1.7) are also necessary. But, unlike what happens in the previous two cases (i.e., for the ellipses $\mathcal{E}_{\mathtt{P}}$ and $\mathcal{E}_{\mathtt{S}}$), now the conics $\mathcal{E}_{\mathtt{I}}$ and $\mathcal{H}_{\mathtt{C}}$ are obtained as the contour of the parallel projection of a suitable one-sheeted hyperboloid centered in O and with axis perpendicular to the plane ω.

1.1 Main Definitions

In the Euclidean space $\mathbb{E}^3$ we fix a plane ω and a system of coordinates such that

$$\omega \stackrel{\mathrm{def}}{=} \Big\{(x,y,z)\in\mathbb{R}^3 \,\Big|\, z=0\Big\}. \tag{1.8}$$

We denote with $O\in\omega$ the origin of coordinates.

Definition 1.1 *Given a plane π and a non-zero vector $\mathbf{w}$, $\mathbf{w}\nparallel\pi$, we say that P,Q are obliquely symmetrical with respect to π, in the direction of $\mathbf{w}$, if*

$$PQ \parallel \mathbf{w} \quad \textit{and} \quad \frac{P+Q}{2}\in\pi.\,^{3} \tag{1.9}$$

Definition 1.2 *Given a non-zero vector*

$$\mathbf{v} = l\mathbf{i}+m\mathbf{j}+n\mathbf{k} \qquad (l,m,n\in\mathbb{R}), \tag{1.10}$$

we denote with $\pi_{\mathbf{v}}$ the plane

$$\pi_{\mathbf{v}}:\ lx+my-nz=0. \tag{1.11}$$

When $\mathbf{v}\nparallel\pi_{\mathbf{v}}$ (i.e., if $l^2+m^2-n^2\neq 0$), we say that P,P' are $\pi_{\mathbf{v}}$–symmetric if P,P' are obliquely symmetrical with respect to the plane $\pi_{\mathbf{v}}$, in the direction of $\mathbf{v}$.

For $\rho>0$, we denote with $\mathscr{H}=\mathscr{H}(\rho)$ be the one-sheeted hyperboloid

$$\mathscr{H}(\rho) \stackrel{\mathrm{def}}{=} \big\{(x,y,z)\in\mathbb{R}^3 \,\big|\, x^2+y^2-z^2=\rho^2\big\}. \tag{1.12}$$

Furthermore, given a point $P\in\mathscr{H}$, we indicate with $T_{\mathscr{H}}(P)$ the tangent plane to $\mathscr{H}$ at P. Namely, if $P=P(x_P,y_P,z_P)$, the plane

$$T_{\mathscr{H}}(P):\ x_Px+y_Py-z_Pz=\rho^2. \tag{1.13}$$

[2] $\mathcal{H}_{\mathtt{C}}$ circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ in the sense of Def. 2.8 below.

[3] With $\frac{P+Q}{2}$ we will indicate the midpoint of the segment PQ.

Definition 1.3 *Let $\mathbf{v}$ be a non-zero vector such that $\mathbf{v} \nparallel \omega$. We denote with*

$$\Pi_{\mathbf{v}} : \mathbb{R}^3 \longrightarrow \omega \tag{1.14}$$

the parallel projection onto ω, in the direction of $\mathbf{v}$. If $\mathbf{v} \nparallel \omega$ and $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, we say that $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is non-degenerate for $\mathscr{H}$ (or, simply, non-degenerate) if $l^2 + m^2 - n^2 \neq 0$. Similarly, we say that $\mathbf{v}$ gives a non-degenerate projection direction.

Noting that the hyperboloid $\mathscr{H}$ is $\pi_{\mathbf{v}}$–symmetric if $\mathbf{v}$ is non-degenerate (see Claim 2.1), we can give the following definition:

Definition 1.4 *Let $OP_1, OP_2, OP_3 \subset \omega$ be three segments which are not contained in a line.*

A non-degenerate parallel projection $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3 if there are a hyperboloid $\mathscr{H} = \mathscr{H}(\rho)$, for some $\rho > 0$, and three points $Q_1, Q_2, Q_3 \in \mathscr{H}$ such that

$$\Pi_{\mathbf{v}}(Q_i) = P_i \quad (1 \leq i \leq 3), \tag{1.15}$$

$$OQ_1 \parallel T_{\mathscr{H}}(Q_2),\ OQ_2 \parallel T_{\mathscr{H}}(Q_3) \ \text{ and } \ OQ_3 \parallel T_{\mathscr{H}}(Q_1'), \tag{1.16}$$

where $Q_1' \in \mathscr{H}$ is $\pi_{\mathbf{v}}$–symmetric to Q_1 in the sense of Def. 1.2 above.

Remark 1.5 *Def. 1.4 is the immediate extension, to the case of the hyperboloid $\mathscr{H}$, of the definition of secondary Pohlke's projection given in [7, Def. 1.2, p.35]. See also Claim A.3.*

Given $P, Q \in \mathscr{H}$, we have that $OQ \parallel T_{\mathscr{H}}(P) \Rightarrow OP \nparallel OQ$.[4] *Therefore, if* (1.16) *holds, $OQ_1 \nparallel OQ_2$, $OQ_2 \nparallel OQ_3$ and $OQ_3 \nparallel OQ_1'$. With condition* (1.16) *we require that the intersections of $\mathscr{H}$ with the planes passing through O, Q_1, Q_2, through O, Q_2, Q_3 and through O, Q_3, Q_1' are three ellipses having as conjugate semi-diameters the pairs (OQ_1, OQ_2), (OQ_2, OQ_3) and (OQ_3, OQ_1') respectively. See Claim 2.29.*

If the segments OP_1, OP_2, OP_3 are not parallel to each other, we can think (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1) as pairs of conjugate semi-diameters of three concentric ellipses.

Definition 1.6 *Given $OP, OQ \subset \omega$, $OP \nparallel OQ$, we denote with $\mathcal{E}_{P,Q}$ the ellipse with OP, OQ as conjugate semi-diameters.*

Then, considering the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$, we give the following definition:

Definition 1.7 *Suppose OP_1, OP_2, OP_3 are non-parallel. A conic $\mathcal{C}$, with center O, is a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3 if one of the following holds:*

- *$\mathcal{C}$ is an ellipse inscribed in $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.*[5]
- *$\mathcal{C}$ is a hyperbola which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.*[5]

To illustrate Def. 1.4 and Def. 1.7, we will give two examples after stating the main results. See Ex. 1.12 and Ex. 1.13 below.

[4] See Rem. 2.28 below.

[5] In other words, we require that $\mathcal{C}$ be tangent to the three ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ and that

a) $\mathcal{C} \subset \mathbf{int}(\mathcal{E}_{P_1,P_2}) \cap \mathbf{int}(\mathcal{E}_{P_2,P_3}) \cap \mathbf{int}(\mathcal{E}_{P_3,P_1})$, if $\mathcal{C}$ is an ellipse;

b) $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1} \subset \mathbf{int}(\mathcal{C})$, if $\mathcal{C}$ is a hyperbola.

Here, given a central conic $\mathcal{C}$ (i.e., an ellipse or a hyperbola) in a plane π, we denote with $\mathbf{int}(\mathcal{C})$ (interior of $\mathcal{C}$) the closure in π of the connected component of $\pi \setminus \mathcal{C}$ containing the center of $\mathcal{C}$. See Defs. 2.7, 2.8 below.

1.2 Main Results

Theorem 1.8 *Suppose the segments OP_1, OP_2, OP_3 are non-parallel. Then the following three properties are equivalent:*

(1) *there is a hyperbolic Pohlke's projection $\Pi_{\mathbf{v}}$ for OP_1, OP_2, OP_3;*

(2) *there is a hyperbolic Pohlke's conic $\mathcal{C}$ for OP_1, OP_2, OP_3;*

(3) $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$ *with h, k satisfying the conditions*

$$f(h,k) \stackrel{\text{def}}{=} (h^2+k^2-1)\big[(h^2-k^2)^2-1\big] \neq 0 \tag{1.17}$$

and

$$g(h,k) \stackrel{\text{def}}{=} (h+k+1)(h+k-1)(h-k+1)(h-k-1) < 0. \tag{1.18}$$

If the above conditions are true, the conic $\mathcal{C}$ is unique and it turns out that $\mathcal{C}$ is an ellipse if $f(h,k) < 0$, while $\mathcal{C}$ is a hyperbola if $f(h,k) > 0$. The projection $\Pi_{\mathbf{v}}$ is unique up to symmetry with respect to the plane ω. [6] *Besides,*

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}) \tag{1.19}$$

where $\mathscr{H}, \pi_{\mathbf{v}}$ satisfy the conditions of Def. 1.2 and Def. 1.4.

If OP_1, OP_2, OP_3 are not contained in a line but two of them are parallel, in particular if one of them vanishes, we need to introduce degenerate ellipses. [7]

Definition 1.9 *If OP, OQ do not both vanish and $OP \parallel OQ$, the degenerate ellipse $\mathcal{E}_{P,Q}$ is the segment MN parallel to OP, OQ such that*

$$|MN|^2 = 4\big(|OP|^2 + |OQ|^2\big) \quad \text{and} \quad \frac{M+N}{2} = O. \tag{1.20}$$

Given a central conic $\mathcal{C}$, with center O, we say that $\mathcal{C}$ circumscribes the degenerate ellipse $\mathcal{E}_{P,Q}$ (or that $\mathcal{E}_{P,Q}$ is inscribed in $\mathcal{C}$) if $M, N \in \mathcal{C}$. [8]

Now we can reformulate Def. 1.7 just saying that:

Definition 1.10 *If OP_1, OP_2, OP_3 are not contained in a line but two of them are parallel, a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3 is a hyperbola, with center O, circumscribing the three (eventually degenerate) ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.*

Using the Def. 1.10, instead of Def. 1.7, we can state the following:

Theorem 1.11 *Suppose OP_1, OP_2, OP_3 are not contained in a line. If two of them are parallel, then there are infinitely many, distinct hyperbolic Pohlke's projections (conics) if these two segments are equal (i.e., congruent), none if they are different.*

[6] In the sense that, in Def. 1.4, the hyperboloid $\mathscr{H}(\rho)$ is unique and the projection direction (represented by the vector $\mathbf{v}$) is unique up to orthogonal symmetry (i.e., the usual symmetry) with respect to ω.

[7] We assume, by convention, that the null segment is parallel to any other segment. Degenerate ellipses were introduced in [2, pp. 372-373]. See also Defs. 3.1, 3.3 of [7].

[8] Note that $M, N \in \mathcal{C} \Rightarrow \mathcal{E}_{P,Q} \subset \mathbf{int}(\mathcal{C})$ See Def. 2.7 below.

1.3 Examples of hyperbolic Pohlke's projections and conics

We conclude the introduction by constructing two examples of hyperbolic Pohlke's projections. Thm. 1.8 will apply in both cases: in the first the hyperbolic Pohlke's conic $\mathcal{C}$ will be an ellipse; in the second $\mathcal{C}$ will be a hyperbola.

To proceed, it is easier to set ρ right away (we take $\rho = 1$) and then work backwards choosing Q_1, Q_2, Q_3 and Q_1' (or, equivalently for (1.22), Q_3') in $\mathscr{H} = \mathscr{H}(\rho)$ so that (1.16) holds. For this, we have only to take into account Claims 2.1, 2.25 and Cor. 2.26. Having done this, and verified that $Q_3Q_3' \nparallel \omega$, the projection direction, i.e., $\mathbf{v} \parallel Q_3Q_3'$, and the points $P_1, P_2, P_3 \in \omega$ are uniquely determined. We recall the following facts: [9]

(1) given $R(x_R, y_R, z_R), S(x_S, y_S, z_S) \in \mathscr{H}$,

$$OR \parallel T_{\mathscr{H}}(S) \quad \Leftrightarrow \quad OS \parallel T_{\mathscr{H}}(R) \quad \Leftrightarrow \quad x_R x_S + y_R y_S - z_R z_S = 0; \tag{1.21}$$

(2) if $R', S' \in \mathscr{H}$ are $\pi_{\mathbf{v}}$–symmetric to $R, S \in \mathscr{H}$ respectively, then

$$OR' \parallel T_{\mathscr{H}}(S) \quad \Leftrightarrow \quad OR \parallel T_{\mathscr{H}}(S'). \tag{1.22}$$

Given $Q_1 \in \mathscr{H}$, applying (1.21) we choose $Q_2, Q_3^* \in \mathscr{H}$ such that $OQ_2, OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$,[10] and $Q_3 \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$. For a), b) of Rem. A.5, we always get $Q_3 \neq Q_3^*$ and we can choose Q_3, Q_3^* such that $Q_3Q_3^* \nparallel \omega$ gives the direction of a non-degenerate projection onto the plane ω. For c) of Rem. A.5, to prevent two of the segments OP_1, OP_2, OP_3 from being parallel, we also require $OQ_3 \nparallel OQ_1$ and, after choosing Q_2 and Q_3,

$$OQ_3^* \nparallel OQ_2 \quad \text{and} \quad \overrightarrow{OQ_3^*} \neq \overrightarrow{OQ_3} - \bar{\lambda}\overrightarrow{OQ_1} \pm \bar{\lambda}\overrightarrow{OQ_2}, \tag{1.23}$$

where $\bar{\lambda} \overset{\text{def}}{=} x_1x_3 + y_1y_3 - z_1z_3$ with $Q_1(x_1, y_1, z_1)$, $Q_3(x_3, y_3, z_3)$. In this way we will be in the hypotheses of Thm. 1.8.

Then, taking $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} \parallel Q_3Q_3^*$, it follows $\mathbf{v}$ is non-degenerate and that Q_3 and Q_3^* are $\pi_{\mathbf{v}}$–symmetric, i.e., $Q_3^* = Q_3'$.[11] By (1.22), we have also $OQ_3 \parallel T_{\mathscr{H}}(OQ_1')$, because $OQ_3' = OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$. The conditions of (1.16) are therefore satisfied and we can define $P_i = \Pi_{\mathbf{v}}(Q_i)$ $(1 \leq i \leq 3)$. For (1.23), OP_1, OP_2, OP_3 are non-parallel, so we can draw the ellipses $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. Finally we trace the contour of the projection of $\mathscr{H}$ into ω:

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}). \tag{1.24}$$

By Cor. 2.11, $\mathcal{C}$ is an ellipse (hyperbola) if $l^2 + m^2 - n^2 < 0$ (> 0). By Thm. 1.8, $\mathcal{C}$ is the unique hyperbolic Pohlke's conic for OP_1, OP_2, OP_3. We refer to Section A.3 for more details.

Example 1.12 *We fix $\rho = 1$ and $Q_1 = (2, 0, -\sqrt{3})$. Taking into account (1.21)–(1.23) we choose $Q_2 = (-\sqrt{3}, \sqrt{2}, 2)$, $Q_3 = (0, \sqrt{2}, 1)$ and $Q_3' = (\frac{3\sqrt{3}}{4}, \frac{5}{4}, -\frac{3}{2})$. In this way the projection direction is given by the vector*

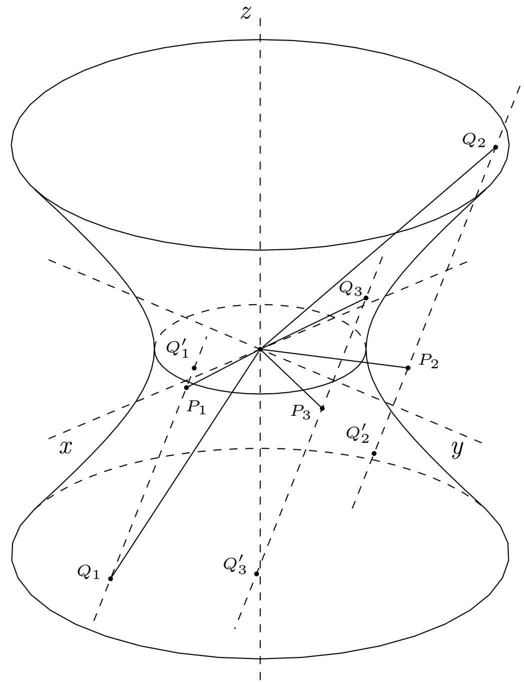
$$\mathbf{v} = \tfrac{3\sqrt{3}}{4}\mathbf{i} + \big(\tfrac{5}{4} - \sqrt{2}\big)\mathbf{j} - \tfrac{5}{2}\mathbf{k}. \tag{1.25}$$

Since $\big(\frac{3\sqrt{3}}{4}\big)^2 + \big(\frac{5}{4} - \sqrt{2}\big)^2 - \big(\frac{5}{2}\big)^2 < 0$, the projection direction is non-degenerate.

[9] See Claim 2.25 and Cor. 2.26 below.

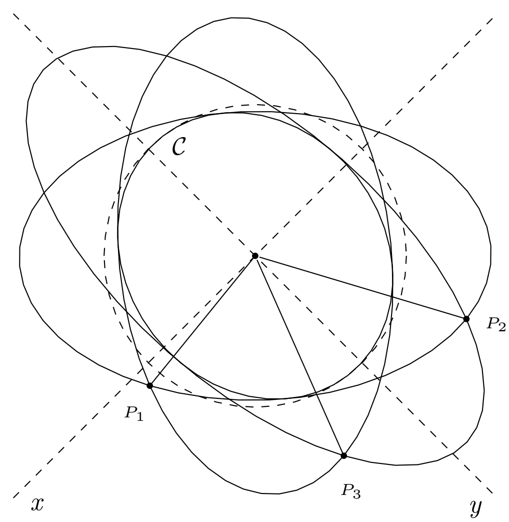
[10] We introduce Q_3^* just to proceed, because we don't have $\mathbf{v}$ yet. We will soon set $Q_3' = Q_3^*$.

[11] See Claim 2.1 below.



Hyperbolic Pohlke's projection of Ex. 1.12 (scaled 0.7).

We find $P_1 = (\frac{11}{10}, \frac{4\sqrt{6}-5\sqrt{3}}{10}, 0)$, $P_2 = (-\frac{2\sqrt{3}}{5}, \frac{5+\sqrt{2}}{5}, 0)$ *and* $P_3 = (\frac{3\sqrt{3}}{10}, \frac{6\sqrt{2}+5}{10}, 0)$. *The segments* OP_1, OP_2, OP_3 *are non-parallel and the hyperbolic Pohlke's conic* $\mathcal{C}$ *is an ellipse.*

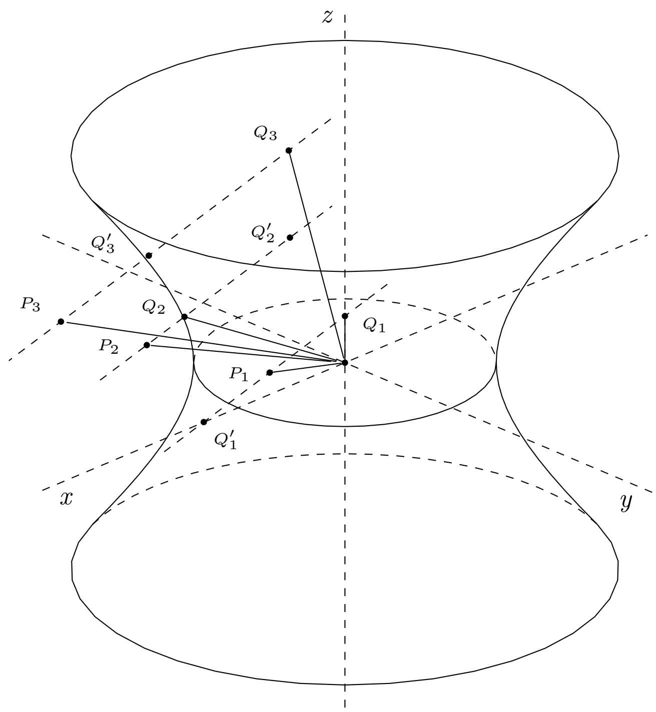


In plane $\{z = 0\}$, *the ellipse* $\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ *inscribed in* $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

Example 1.13 *We fix* $\rho = 1$ *and* $Q_1 = (1, 1, 1)$. *Taking into account* (1.21)–(1.23) *we choose* $Q_2 = (1, -\frac{1}{2}, \frac{1}{2})$, $Q_3 = (\frac{5}{2}, \frac{79}{40}, \frac{121}{40})$ *and* $Q_3' = (\frac{3}{2}, -\frac{1}{3}, \frac{7}{6})$. *In this way the projection direction is given by the vector*

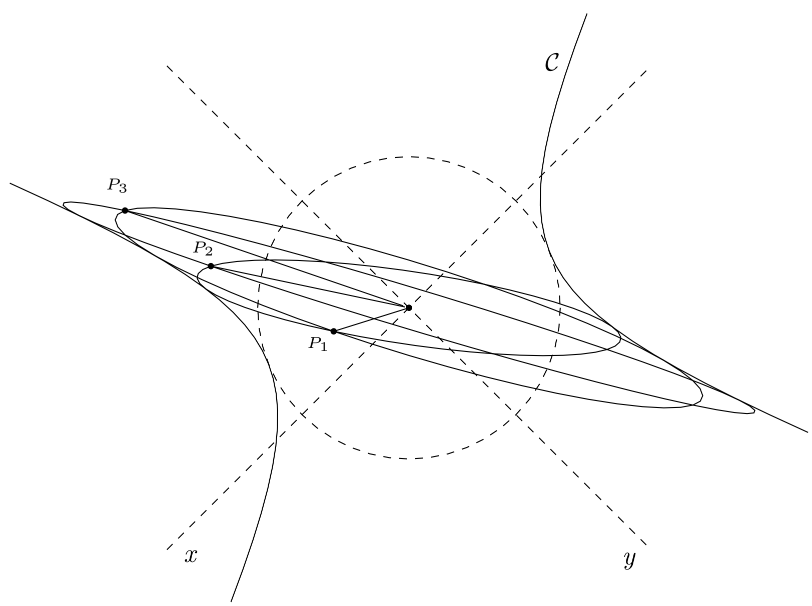
$$\mathbf{v} = \mathbf{i} + \tfrac{277}{120}\mathbf{j} + \tfrac{223}{120}\mathbf{k}. \tag{1.26}$$

Since $1+\left(\frac{277}{120}\right)^2-\left(\frac{223}{120}\right)^2>0$*, the projection direction is non-degenerate.*



Hyperbolic Pohlke's projection of Ex. 1.13.

We find $P_1=(\frac{103}{223},-\frac{54}{223},0)$*,* $P_2=(\frac{163}{223},-\frac{250}{223},0)$ *and* $P_3=(\frac{389}{446},-\frac{795}{446},0)$*. The segments* OP_1*,* OP_2*,* OP_3 *are non-parallel and the hyperbolic Pohlke's conic* $\mathcal{C}$ *is a hyperbola.*



In plane $\{z=0\}$*, the hyperbola* $\mathcal{C}=\Pi_{\mathbf{v}}(\mathscr{H}\cap\pi_{\mathbf{v}})$ *circumscribing* $\mathcal{E}_{P_1,P_2}$*,* $\mathcal{E}_{P_2,P_3}$*,* $\mathcal{E}_{P_3,P_1}$*.*

2 Some basic geometric facts

We shall prove here a number of facts about the one-sheeted hyperboloid $\mathscr{H} = \mathscr{H}(\rho)$ defined in (1.12). We start with some symmetry properties.

Claim 2.1 *Let $\pi_{\mathbf{v}}$ be the plane introduced in Def. 1.2 and let us suppose that $l^2 + m^2 - n^2 \neq 0$. Then $\mathscr{H}$ is $\pi_{\mathbf{v}}$–symmetric (i.e., $P \in \mathscr{H} \Rightarrow P' \in \mathscr{H}$).*

Proof. Indeed, let r be any line parallel to $\mathbf{v}$, that is,

$$r : \begin{cases} x = x_o + lt \\ y = y_o + mt \\ z = z_0 + nt \end{cases} \qquad (t \in \mathbb{R}), \text{ for a suitable } P(x_o, y_o, z_o). \tag{2.1}$$

Introducing the expressions (2.1) into the equation of $\mathscr{H}$, we see that the points of $r \cap \mathscr{H}$ are determined by the real solutions of

$$(l^2 + m^2 - n^2)t^2 + 2(lx_o + my_o - nz_o)t + x_o^2 + y_o^2 - z_o^2 = \rho^2. \tag{2.2}$$

Since $l^2 + m^2 - n^2 \neq 0$, equation (2.2) is of second degree with roots t_1, t_2 such that

$$\frac{t_1 + t_2}{2} = -\frac{lx_o + my_o - nz_o}{l^2 + m^2 - n^2}\,. \tag{2.3}$$

Now, if $P \in \mathscr{H}$, the solutions of (2.2) are

$$t_1 = 0 \quad \text{and} \quad t_2 = -2\,\frac{lx_o + my_o - nz_o}{l^2 + m^2 - n^2}\,. \tag{2.4}$$

Hence $r \cap \mathscr{H} = \{P(t_1), P(t_2)\}$ with $P(t_1) = P$ and $P(t_2)$ such that

$$\frac{P(t_1) + P(t_2)}{2} = P\left(\frac{t_1 + t_2}{2}\right) \in \pi_{\mathbf{v}}, \tag{2.5}$$

because of (2.1), (2.3). Thus $P(t_2) = P'$. $\square$

Remark 2.2 *From the proof of Claim 2.1 one can also see that r is tangent to $\mathscr{H}$ at P iff $P \in \mathscr{H} \cap \pi_{\mathbf{v}}$. In fact, if $P \in \mathscr{H}$, we have $t_1 = t_2 \Leftrightarrow lx_o + my_o - nz_o = 0$.*

Definition 2.3 *Let $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ with $l^2 + m^2 - n^2 \neq 0$. We indicate with $\mathtt{S}_{\mathbf{v}}$ the map associated to the oblique symmetry with respect to $\pi_{\mathbf{v}}$, in the direction of $\mathbf{v}$. That is the map*

$$P(x, y, z) \xrightarrow{\mathtt{S}_{\mathbf{v}}} P'(x', y', z'), \tag{2.6}$$

given by

$$\mathtt{S}_{\mathbf{v}}(x, y, z) = (x - 2\lambda l, y - 2\lambda m, z - 2\lambda n) \quad \textit{with} \quad \lambda = \frac{lx + my - nz}{l^2 + m^2 - n^2}. \quad \square \tag{2.7}$$

Note that, according to Def. 2.3, $\mathtt{S}_{\mathbf{k}}$ represents the orthogonal symmetry (i.e., the usual symmetry) with respect to the plane ω. Besides, we observe that

Remark 2.4 *We can also get Claim 2.1 directly from the oblique symmetry $\mathtt{S}_{\mathbf{v}}$ introduced in Def. 2.3. Indeed, it is easy to see that $\mathtt{S}_{\mathbf{v}}(P) \in \mathscr{H}$ iff $P \in \mathscr{H}$.*

2.1 The intersection of $\mathscr{H}$ with the plane $\pi_{\mathbf{v}}$

If $\pi_{\mathbf{v}}$ is the plane introduced in Def. 1.2, it is clear that $\mathscr{H} \cap \pi_{\mathbf{v}}$ is non-empty, symmetric with respect to O and such that $O \notin \mathscr{H} \cap \pi_{\mathbf{v}}$. Thus $\mathscr{H} \cap \pi_{\mathbf{v}}$ must be a central conic in $\pi_{\mathbf{v}}$ with center O, if it is non-degenerate. [12] On the other hand, if $\mathscr{H} \cap \pi_{\mathbf{v}}$ is degenerate, then it is a pair of distinct, parallel lines which are symmetric with respect to O. More precisely,

Claim 2.5 *Let $\pi_{\mathbf{v}}$ be the plane introduced in Def. 1.2, then*

(1) $\mathscr{H} \cap \pi_{\mathbf{v}}$ is an ellipse $\Leftrightarrow$ $l^2 + m^2 - n^2 < 0$. [13]

(2) $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a pair of distinct, parallel lines $\Leftrightarrow$ $l^2 + m^2 - n^2 = 0$.

(3) $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a hyperbola $\Leftrightarrow$ $l^2 + m^2 - n^2 > 0$.

Proof. $\Leftarrow$ To begin with, let us suppose $n = 0$. Then we have $l^2 + m^2 - n^2 > 0$ and

$$\pi_{\mathbf{v}} : \, lx + my = 0.$$

From the identity

$$(l^2 + m^2)(x^2 + y^2) \equiv (lx + my)^2 + (mx - ly)^2, \tag{2.8}$$

we deduce that $\mathscr{H} \cap \pi_{\mathbf{v}}$ is given by the points $P(x, y, z) \in \pi_{\mathbf{v}}$ such that

$$\frac{(mx - ly)^2}{l^2 + m^2} - z^2 = \rho^2. \tag{2.9}$$

But (2.9) represents a hyperbola in $\pi_{\mathbf{v}}$, because $h = \frac{mx-ly}{\sqrt{l^2+m^2}}$ and $k = z$ may be considered as coordinates in $\pi_{\mathbf{v}}$. Next, let us suppose $n \neq 0$. In this case the coordinates of the points of $\pi_{\mathbf{v}}$ satisfy the relation

$$z = \frac{lx + my}{n}. \tag{2.10}$$

Hence $\mathscr{H} \cap \pi_{\mathbf{v}}$ is given by the points $P(x, y, z) \in \pi_{\mathbf{v}}$ such that

$$(n^2 - l^2)x^2 + (n^2 - m^2)y^2 - 2lmxy = n^2\rho^2. \tag{2.11}$$

Equation (2.11) (with the condition $z = 0$) defines a conic $\mathcal{C}$, with center O, in the plane ω. Namely, $\mathcal{C}$ is an ellipse if $l^2 + m^2 - n^2 < 0$, while $\mathcal{C}$ is a hyperbola if $l^2 + m^2 - n^2 > 0$. Noting that $\mathscr{H} \cap \pi_{\mathbf{v}}$ is the affine image of $\mathcal{C}$ via the parallel projection $\mathtt{T} : \omega \to \pi_{\mathbf{v}}$,

$$(x, y) \xrightarrow{\mathtt{T}} \left(x, y, \frac{lx + my}{n}\right), \tag{2.12}$$

by Thm. A.1 we deduce that $\mathscr{H} \cap \pi_{\mathbf{v}}$ is an ellipse or a hyperbola depending on whether the quantity $l^2 + m^2 - n^2$ is < 0 or > 0, respectively. [14] Finally, let us suppose $n \neq 0$ and $l^2 + m^2 - n^2 = 0$. In this last case (2.11) factorizes as

$$(mx - ly + n\rho)(mx - ly - n\rho) = 0, \tag{2.13}$$

[12] That is, a non-degenerate conic with center, i.e., an ellipse or a hyperbola.

[13] We will distinguish between circles and ellipses only when strictly necessary. In this case it is not difficult to show that $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a circle $\Leftrightarrow$ $l = m = 0$.

[14] As stated in Thm. A.1 of the Appendix, an affine transformation between two planes $\pi, \tilde{\pi} \subset \mathbb{E}^3$ send a conic to a conic of the same type. This applies, in particular, to parallel projections between planes.

because $n^2 - l^2 = m^2$ and $n^2 - m^2 = l^2$. Thus $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a pair of distinct, parallel lines which are symmetric with respect to O.
$\Rightarrow$ The reverse implication is now an immediate consequence of the fact that by proving $\Leftarrow$ we have exhausted all possible cases for the sign of the quantity $l^2 + m^2 - n^2$. $\square$

For the sake of brevity, we will later say that:

Definition 2.6 *$\mathcal{C}$ is an admissible conic if $\mathcal{C}$ is a central conic centered at O, or a pair of distinct, parallel lines which are symmetric with respect to O.*

2.2 The projection of $\mathscr{H}$ into the plane $\pi_{\mathbf{v}}$

We will adopt the following terminology:

Definition 2.7 *Let $\mathcal{C} \subset \pi$ be a central conic in a plane π, i.e., an ellipse or a hyperbola.*

(1) We denote with $\mathbf{int}(\mathcal{C})$ (interior of $\mathcal{C}$) the closure in π of the connected component of $\pi \setminus \mathcal{C}$ containing the center of $\mathcal{C}$.

(2) We denote also with $\mathbf{ext}(\mathcal{C})$ (exterior of $\mathcal{C}$) the closure in π of $\pi \setminus \mathbf{int}(\mathcal{C})$.

Definition 2.8 *Given two concentric central conics $\mathcal{C}_1, \mathcal{C}_2 \subset \omega$, we will say that $\mathcal{C}_1$ is inscribed in $\mathcal{C}_2$ (or, equivalently, that $\mathcal{C}_2$ circumscribes $\mathcal{C}_1$) if $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent and $\mathcal{C}_1 \subset \mathbf{int}(\mathcal{C}_2)$.*

Remark 2.9 *As it is known, in an appropriate coordinate system (say $\mathbf{x}, \mathbf{y}$) a central conic $\mathcal{C}$ has the simple equation $\lambda \mathbf{x}^2 + \mu \mathbf{y}^2 = 1$, with $\lambda > 0$ and $\mu \neq 0$. Then, we have*

$$\mathbf{int}(\mathcal{C}) = \big\{P(\mathbf{x},\mathbf{y}) \in \zeta \,\big|\, \lambda \mathbf{x}^2 + \mu \mathbf{y}^2 \leq 1\big\},$$
$$\mathbf{ext}(\mathcal{C}) = \big\{P(\mathbf{x},\mathbf{y}) \in \zeta \,\big|\, \lambda \mathbf{x}^2 + \mu \mathbf{y}^2 \geq 1\big\}. \quad \square$$

If $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ provides a non-degenerate projection direction (Def. 1.3), i.e.,

$$l^2 + m^2 - n^2 \neq 0, \tag{2.14}$$

we may consider the parallel projection $\widetilde{\Pi}_{\mathbf{v}} : \mathbb{R}^3 \to \pi_{\mathbf{v}}$ in the direction of $\mathbf{v}$; that is

$$\widetilde{\Pi}_{\mathbf{v}}(x,y,z) \stackrel{\text{def}}{=} (x - \lambda l, y - \lambda m, z - \lambda n) \quad \text{with} \quad \lambda = \frac{lx + my - nz}{l^2 + m^2 - n^2}. \tag{2.15}$$

Applying Claim 2.5, we have:

Claim 2.10 *Let $\widetilde{\Pi}_{\mathbf{v}} : \mathbb{R}^3 \to \pi_{\mathbf{v}}$ be the projection defined by (2.15), then*

(a) *$\widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) = \mathbf{ext}(\mathscr{H} \cap \pi_{\mathbf{v}})$ if $\mathscr{H} \cap \pi_{\mathbf{v}}$ is an ellipse, i.e., if $l^2 + m^2 - n^2 < 0$.*

(b) *$\widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) = \mathbf{int}(\mathscr{H} \cap \pi_{\mathbf{v}})$ if $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a hyperbola, i.e., if $l^2 + m^2 - n^2 > 0$.*

Proof. Indeed, given $P(x_o, y_o, z_o) \in \pi_{\mathbf{v}}$, we have that

$$P \in \widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) \quad \Longleftrightarrow \quad \text{equation (2.2) has a real solution.}$$

Since $lx_o + my_o - nz_o = 0$ in $\pi_{\mathbf{v}}$, from (2.2) we see that:

($\mathtt{a}'$) $P \in \widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) \;\Leftrightarrow\; x_o^2 + y_o^2 - z_o^2 \geq \rho^2$, if $\mathscr{H} \cap \pi_{\mathbf{v}}$ is an ellipse.

($\mathtt{b}'$) $P \in \widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) \;\Leftrightarrow\; x_o^2 + y_o^2 - z_o^2 \leq \rho^2$, if $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a hyperbola.

Supposing for the moment $n \neq 0$ and using (2.10), we see that:

($\mathtt{a}''$) $x_o^2 + y_o^2 - z_o^2 \geq \rho^2 \;\Leftrightarrow\; (n^2 - l^2)x_o^2 + (n^2 - m^2)y_o^2 - 2mlx_oy_o \geq n^2\rho^2$,

($\mathtt{b}''$) $x_o^2 + y_o^2 - z_o^2 \leq \rho^2 \;\Leftrightarrow\; (n^2 - l^2)x_o^2 + (n^2 - m^2)y_o^2 - 2mlx_oy_o \leq n^2\rho^2$,

depending on whether $\mathscr{H} \cap \pi_{\mathbf{v}}$ is an ellipse or a hyperbola, respectively. In other words, given $P(x_o, y_o, z_o) \in \pi_{\mathbf{v}}$, we deduce that:

($\mathtt{a}'''$) $P \in \widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) \;\Leftrightarrow\; (x_o, y_o)$ is exterior to the ellipse $\mathcal{E} \subset \omega$, with

$$\mathcal{E}\colon (n^2 - l^2)x^2 + (n^2 - m^2)y^2 - 2mlxy = n^2\rho^2,$$

($\mathtt{b}'''$) $P \in \widetilde{\Pi}_{\mathbf{v}}(\mathscr{H}) \;\Leftrightarrow\; (x_o, y_o)$ is interior to the hyperbola $\mathcal{H} \subset \omega$, with

$$\mathcal{H}\colon (n^2 - l^2)x^2 + (n^2 - m^2)y^2 - 2mlxy = n^2\rho^2,$$

depending on whether $\mathscr{H} \cap \pi_{\mathbf{v}}$ is respectively an ellipse or a hyperbola.
To conclude, it is now sufficient to observe that in any case the plane $\pi_{\mathbf{v}}$ is the affine image, via the parallel projection $\mathtt{T} : \omega \to \pi_{\mathbf{v}}$ in (2.12), of the plane ω and that, by (2.10)$-$(2.11), the same transformation maps the conic $\mathcal{E} \subset \omega$, or $\mathcal{H} \subset \omega$, onto $\mathscr{H} \cap \pi_{\mathbf{v}}$. Hence we also have:

($\mathtt{a}''''$) (x_o, y_o) is exterior to the ellipse $\mathcal{E} \;\Leftrightarrow\; P$ is exterior to the ellipse $\mathscr{H} \cap \pi_{\mathbf{v}}$,

($\mathtt{b}''''$) (x_o, y_o) is interior to the hyperbola $\mathcal{H} \;\Leftrightarrow\; P$ is interior to the hyperbola $\mathscr{H} \cap \pi_{\mathbf{v}}$,

depending on whether $\mathscr{H} \cap \pi_{\mathbf{v}}$ is respectively an ellipse or a hyperbola.[14]

Finally, let us suppose $n = 0$. In this case $l^2 + m^2 - n^2 > 0$, thus $\mathscr{H} \cap \pi_{\mathbf{v}}$ is a hyperbola in $\pi_{\mathbf{v}}$. Given $P = P(x_o, y_o, z_o) \in \pi_{\mathbf{v}}$, using the identity (2.8) we can rewrite the condition ($\mathtt{b}'$), that is, $x_o^2 + y_o^2 - z_o^2 \leq \rho^2$, in the form

$$(mx_o - ly_o)^2 - (l^2 + m^2)z_o^2 \leq (l^2 + m^2)\rho^2, \tag{2.16}$$

because $P \in \pi_{\mathbf{v}} \;\Leftrightarrow\; lx_o + my_o = 0$. Then, introducing in $\pi_{\mathbf{v}}$ the coordinates $h = \frac{mx - ly}{\sqrt{l^2 + m^2}}$ and $k = z$, we immediately see that condition (2.16) is equivalent to

$$h_o^2 - k_o^2 \leq \rho^2 \quad \text{with} \quad h_o = h(x_o, y_o),\; k_o = z_o. \tag{2.17}$$

This, in turn, is equivalent to saying that $P \in \pi_{\mathbf{v}}$ is interior to the hyperbola $\mathscr{H} \cap \pi_{\mathbf{v}}$. In fact, with the coordinates (h, k), the equation of $\mathscr{H} \cap \pi_{\mathbf{v}}$ is exactly $h^2 - k^2 = \rho^2$, as one can see from the first part of the proof of Claim 2.5. $\square$

2.3 The projection of $\mathscr{H}$ and $\mathscr{H} \cap \pi_{\mathbf{v}}$ into the plane ω

To continue we suppose
$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} \quad \text{with} \quad n \neq 0. \tag{2.18}$$
We can therefore define the projection $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ in the direction of $\mathbf{v}$. Namely
$$\Pi_{\mathbf{v}}(x, y, z) \stackrel{\text{def}}{=} \left(x - \frac{l}{n} z,\ y - \frac{m}{n} z,\ 0 \right). \tag{2.19}$$

Assuming also $l^2 + m^2 - n^2 \neq 0$ the restriction of $\Pi_{\mathbf{v}}$ to the plane $\pi_{\mathbf{v}} : lx + my - nz = 0$ is an affine transformation from $\pi_{\mathbf{v}}$ to ω. Noting that $\Pi_{\mathbf{v}}(O) = O$, taking into account Def. 1.3 and Claim 2.5, we easily have the following:

Corollary 2.11 *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate, then*

(1) $\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ is an ellipse centered at O $\Leftrightarrow$ $l^2 + m^2 - n^2 < 0$. [15]

(2) $\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ is a hyperbola centered at O $\Leftrightarrow$ $l^2 + m^2 - n^2 > 0$.

We can then give the following definition:

Definition 2.12 *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate. We denote with $\mathscr{C}_{\mathbf{v}}$ the conic*
$$\mathscr{C}_{\mathbf{v}} \stackrel{\text{def}}{=} \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}). \tag{2.20}$$

Let us note that
$$\Pi_{\mathbf{v}} = \Pi_{\mathbf{v}} \circ \widetilde{\Pi}_{\mathbf{v}},$$
if $\mathbf{v}$ is non-degenerate. Then, from Claim 2.10, we have:

Corollary 2.13 *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate. Then*

(1) $\Pi_{\mathbf{v}}(\mathscr{H}) = \mathbf{ext}(\mathscr{C}_{\mathbf{v}})$ if $\mathscr{C}_{\mathbf{v}}$ is an ellipse, i.e., if $l^2 + m^2 - n^2 < 0$.

(2) $\Pi_{\mathbf{v}}(\mathscr{H}) = \mathbf{int}(\mathscr{C}_{\mathbf{v}})$ if $\mathscr{C}_{\mathbf{v}}$ is a hyperbola, i.e., if $l^2 + m^2 - n^2 > 0$.

Remark 2.14 *When $l^2 + m^2 - n^2 = 0$ it is easy to see that*
$$\Pi_{\mathbf{v}}(\mathscr{H}) \ = \ \omega \setminus \big\{ (x, y, 0) \,\big|\, lx + my = 0,\ x^2 + y^2 \neq \rho^2 \big\}. \tag{2.21}$$
This follows from (2.2) with $P \in \omega$ and $l^2 + m^2 - n^2 = 0$. Namely, the equation
$$2(lx_o + my_o)t + x_o^2 + y_o^2 = \rho^2. \tag{2.22}$$
Indeed, from (2.22) we can see that $r \cap \mathscr{H} = \emptyset$ iff $lx_o + my_o = 0$ and $x_o^2 + y_o^2 \neq \rho^2$. □

Remembering that $\mathscr{H} = \mathscr{H}(\rho)$, it is also easy to see that:

[15] It follows that $\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ is a circle $\Leftrightarrow$ $l = m = 0$. See Claim 2.15 and Remark 2.17.

Claim 2.15 *Suppose $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is non-degenerate. If l, m are not both 0, then $\mathscr{C}_{\mathbf{v}} \subset \omega$ is a central conic with center O and major/transverse axis orthogonal to $\mathbf{v}$. More precisely,*

1) If $l^2 + m^2 - n^2 < 0$, then $\mathscr{C}_{\mathbf{v}}$ is an ellipse, centered at O, with semi axes a, b such that

$$a = \rho, \qquad b^2 = \rho^2\,\frac{n^2 - l^2 - m^2}{n^2}\,.$$

2) If $l^2+m^2-n^2 > 0$, then $\mathscr{C}_{\mathbf{v}}$ is a hyperbola with transverse semi-axis a and conjugate semi-axis b such that

$$a = \rho, \qquad b^2 = \rho^2\,\frac{l^2 + m^2 - n^2}{n^2}\,.$$

If $l = m = 0$ the conic $\mathscr{C}_{\mathbf{v}} \subset \omega$ is merely the circle with center O and radius ρ.

Proof. Since $\mathscr{H}$ is invariant under rotations about the z-axis, by rotating the projection direction (i.e., the vector $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$) about the z-axis we simply rotate the conic $\mathscr{C}_{\mathbf{v}} \subset \omega$ around the origin O. This means that in proving 1) and 2) of Claim 2.15 we can assume

$$\mathbf{v} = \lambda\mathbf{j} + n\mathbf{k} \quad \text{with} \quad \lambda = \sqrt{l^2 + m^2} \tag{2.23}$$

such that $\lambda^2 - n^2 \neq 0$. Then, from (2.11) with $l = 0$ and $m = \lambda$, the intersection $\mathscr{H} \cap \pi_{\mathbf{v}}$ is given by the points $P = (x, y, z) \in \pi_{\mathbf{v}}$ such that

$$n^2x^2 + (n^2 - \lambda^2)y^2 = n^2\rho^2. \tag{2.24}$$

By (2.23) we have $\pi_{\mathbf{v}} : \lambda y - nz = 0$. This means that $P(x, y, z) \in \mathscr{H} \cap \pi_{\mathbf{v}}$ if and only if

$$x^2 + \left(\frac{n^2 - \lambda^2}{n^2}\right) y^2 = \rho^2, \quad z = \frac{\lambda}{n} y. \tag{2.25}$$

With $\mathbf{v}$ as in (2.23) and $P(x, y, z)$ such that $z = \frac{\lambda}{n} y$, from (2.19) we find

$$\Pi_{\mathbf{v}}(x, y, z) \;=\; \left(x,\ \frac{n^2 - \lambda^2}{n^2}\, y,\ 0\right) \stackrel{\text{def}}{=} (\bar{x}, \bar{y}, \bar{z}). \tag{2.26}$$

Hence the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the points of $\mathscr{C}_{\mathbf{v}} \subset \omega$ satisfy

$$\bar{x}^2 + \left(\frac{n^2}{n^2 - \lambda^2}\right) \bar{y}^2 = \rho^2, \quad \bar{z} = 0. \tag{2.27}$$

For $\lambda \neq 0$, we deduce that

- If $\lambda^2 - n^2 < 0$, then $\mathscr{C}_{\mathbf{v}}$ is an ellipse with major semi-axis $a = \rho$ and minor semi-axis b such that $b^2 = \rho^2\,\frac{n^2-\lambda^2}{n^2}$. The major semi-axis, being along the x-axis, is orthogonal to the direction of projection, i.e., $\mathbf{v}$ given by (2.23).
- If $\lambda^2 - n^2 > 0$, then $\mathscr{C}_{\mathbf{v}}$ is a hyperbola with transverse semi-axis $a = \rho$ and conjugate semi-axis b such that $b^2 = \rho^2\,\frac{\lambda^2-n^2}{n^2}$. The transverse semi-axis, being along the x-axis, is orthogonal to the direction of projection given by (2.23).

Finally, when $\lambda = 0$, the conclusion (which formally follows from (2.27) with $\lambda = 0$) is immediate because $\pi_{\mathbf{v}} = \pi_{\mathbf{k}} = \omega$. $\square$

Remark 2.16 *From* (2.25)–(2.26) *it also follows that if*

$$\mathbf{v} = \lambda \mathbf{j} + n\mathbf{k} \quad \textit{with} \quad \lambda^2 - n^2 = 0, \tag{2.28}$$

then the points $(\bar{x}, \bar{y}, 0)$ *of* $\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ *satisfy* $\bar{x}^2 = \rho^2$, $\bar{y} = 0$. *Hence* $\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ *reduces to the pair* $(\pm\rho, 0, 0)$. *For general* $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, *such that* $l^2 + m^2 - n^2 = 0$, *we find*

$$\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}) = \pm \left(\frac{m\rho}{\sqrt{l^2 + m^2}}, \frac{-l\rho}{\sqrt{l^2 + m^2}}, 0 \right). \quad \square \tag{2.29}$$

Remark 2.17 *From Claim 2.15 we can see that given a central conic* $\mathcal{C} \subset \omega$, *with center* O, *there are a unique* $\rho > 0$ *and, up to symmetry with respect to* ω, *a unique projection direction (represented by the vector* $\mathbf{v}$*) such that*

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}) \stackrel{\text{def}}{=} \mathscr{C}_{\mathbf{v}}.$$

Indeed, ρ *must be equal to the major/transverse semi-axis of* $\mathcal{C}$ *(or the radius, if* $\mathcal{C}$ *is a circle). As for the direction projection, we have:*

- *If* $\mathcal{C}$ *is a circle, then the projection direction is given by the vector* $\mathbf{v} = \mathbf{k}$.
- *If* $\mathcal{C}$ *is an ellipse with semi-axes* OV, OW *such that* $|OV| < |OW|$ *and* $\overrightarrow{OV} = p\mathbf{i} + q\mathbf{j}$, *then* $\rho = |OW|$ *and the projection direction is given by the vectors*

$$\mathbf{v} = \delta p\mathbf{i} + \delta q\mathbf{j} \pm \mathbf{k} \quad \textit{with} \quad \delta = \sqrt{\frac{\rho^2 - p^2 - q^2}{\rho^2(p^2 + q^2)}}. \tag{2.30}$$

- *If* $\mathcal{C}$ *is a hyperbola with conjugate and transverse semi-axes* OV, OW *respectively and if* $\overrightarrow{OV} = p\mathbf{i} + q\mathbf{j}$, *then* $\rho = |OW|$ *and the projection direction is given by the vectors*

$$\mathbf{v} = \delta p\mathbf{i} + \delta q\mathbf{j} \pm \mathbf{k} \quad \textit{with} \quad \delta = \sqrt{\frac{\rho^2 + p^2 + q^2}{\rho^2(p^2 + q^2)}}. \quad \square \tag{2.31}$$

In addition, it is immediate that:

Claim 2.18 *In all three cases of Rem. 2.17 the vector* $\mathbf{v}$ *is non-degenerate.* [16]

Moreover, recalling the Defs. 1.2, 1.3 and Def. 2.6, we have:

Claim 2.19 *Let* $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *be non-degenerate.*

[16] This is obvious in view of Cor. 2.11 and Rem. 2.16. But, setting $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, in the three cases of Rem. 2.17 it is also easy to see that:

$$l^2 + m^2 - n^2 = \begin{cases} -1 & \text{if } \mathcal{C} \text{ is a circle} \\ -\frac{p^2+q^2}{\rho^2} & \text{if } \mathcal{C} \text{ is an ellipse} \\ \frac{p^2+q^2}{\rho^2} & \text{if } \mathcal{C} \text{ is a hyperbola} \end{cases}$$

1) If $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$ is an admissible conic tangent to $\mathscr{C}_{\mathbf{v}}$, then there are $\pi_{\mathbf{v}}-$symmetric planes π, π' through the origin O such that $\pi, \pi' \nparallel \mathbf{v}$ and

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi) = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi'). \tag{2.32}$$

2) Let $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$ be an ellipse or a hyperbola tangent to $\mathscr{C}_{\mathbf{v}}$, and let (OP_1, OP_2) be a pair of conjugate semi-diameters of $\mathcal{C}$.[17] Then there are $Q_1, Q_1', Q_2, Q_2' \in \mathscr{H}$ such that $\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathscr{H} = \{Q_1, Q_1'\}$, $\Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathscr{H} = \{Q_2, Q_2'\}$ and (OQ_1, OQ_2), (OQ_1', OQ_2') are conjugate semi-diameters of the conics $\mathscr{H} \cap \pi$ and $\mathscr{H} \cap \pi'$, respectively.

3) Conversely, if π is a plane through the origin O such that $\pi \nparallel \mathbf{v}$ and $\mathscr{H} \cap \pi_{\mathbf{v}} \cap \pi \neq \emptyset$, then $\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi)$ is an admissible conic, tangent to $\mathscr{C}_{\mathbf{v}}$.

Proof. 1) Let $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$ be an admissible conic tangent to $\mathscr{C}_{\mathbf{v}}$ at X_1 and let

$$t \text{ be the common tangent of } \mathcal{C} \text{ and } \mathscr{C}_{\mathbf{v}} \text{ at } X_1. \tag{2.33}$$

Besides, let $X_2 \in \mathcal{C}$ such that $OX_1 \nparallel OX_2$. Since we assume $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$, we clearly have

$$X_1, X_2 \in \Pi_{\mathbf{v}}(\mathscr{H}). \tag{2.34}$$

Thus there are $Y_1 \in \mathscr{H} \cap \pi_{\mathbf{v}}$ and $Y_2 \in \mathscr{H}$ such that

$$\Pi_{\mathbf{v}}(Y_1) = X_1, \ \Pi_{\mathbf{v}}(Y_2) = X_2 \quad \text{and} \quad OY_1 \nparallel OY_2. \tag{2.35}$$ [18]

To proceed, let π be the plane through the points O, Y_1, Y_2. It is clear that $\pi \nparallel \mathbf{v}$, otherwise we would have $OX_1 = \Pi_{\mathbf{v}}(OY_1) \parallel \Pi_{\mathbf{v}}(OY_2) = OX_2$. Hence the restriction

$$\Pi_{\mathbf{v}}\Big|_{\pi} : \pi \longrightarrow \omega \quad \text{defines an affine transformation.} \tag{2.36}$$

By Claim 2.5 (with π instead of $\pi_{\mathbf{v}}$)

$$\mathscr{H} \cap \pi \tag{2.37}$$

is an *admissible* conic. Then, by (2.36),

$$\mathcal{Q} \stackrel{\text{def}}{=} \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi)$$

is also an *admissible* conic.[14] Furthermore, by (2.33) and Cor. 2.13,

$$X_1 \in \mathcal{Q} \quad \text{and} \quad \mathcal{Q} \subset \Pi_{\mathbf{v}}(\mathscr{H}) \quad \Longrightarrow \quad \mathcal{Q} \text{ has tangent } t \text{ at } X_1.$$ [19]

[17] If $\mathcal{C}$ is a hyperbola then OP_1, OP_2 are a transverse semi-diameter and the corresponding "imaginary" conjugate semi-diameter in the sense of Rem. A.2 in the Appendix.

[18] Note that Y_1 is unique. In fact, assuming $X_1 \in \mathscr{C}_{\mathbf{v}}$, the line through X_1 and parallel to $\mathbf{v}$ is tangent to $\mathscr{H}$ at a point of $\mathscr{H} \cap \pi_{\mathbf{v}}$. Y_2 is unique up to $\pi_{\mathbf{v}}$–symmetry, because $\Pi_{\mathbf{v}}^{-1}(X_2) \cap \mathscr{H} = \{Y_2, Y_2'\}$ with Y_2, Y_2' $\pi_{\mathbf{v}}-$symmetric; furthermore, $Y_2 = Y_2' \Leftrightarrow X_2 \in \mathscr{C}_{\mathbf{v}}$. See Claim 2.1 and Rem. 2.2 above. Finally, being $\Pi_{\mathbf{v}}$ an affine transformation, it follows that $OX_1 \nparallel OX_2 \Rightarrow OY_1 \nparallel OY_2$.

[19] Indeed, being an admissible conics (Def. 2.6), $\mathcal{Q}$ has tangent at all its points. Let t_1 be the tangent of $\mathcal{Q}$ at X_1. Since $X_1 \in \mathscr{C}_{\mathbf{v}}$ and $\mathscr{C}_{\mathbf{v}}$ has tangent t at X_1, if $t_1 \neq t$ then $\mathcal{Q} \not\subset \mathbf{int}(\mathscr{C}_{\mathbf{v}})$ and $\mathcal{Q} \not\subset \mathbf{ext}(\mathscr{C}_{\mathbf{v}})$ at the same time. This fact contradicts Cor. 2.13, because $\mathcal{Q} \subset \Pi_{\mathbf{v}}(\mathscr{H})$.

This means that $\mathcal{Q}$ has in common with $\mathcal{C}$ the point X_1, the tangent t at X_1 and a second point X_2 such that $OX_1 \nparallel OX_2$. Since $\mathcal{C}$ and $\mathcal{Q}$ are both symmetric with respect to the origin O (and do not pass through O), it follows that $\mathcal{C} = \mathcal{Q} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi)$.[20] Furthermore, taking into account that $\mathscr{H}$ is $\pi_{\mathbf{v}}-$symmetric, if the plane π' is $\pi_{\mathbf{v}}-$symmetric to π we also find

$$\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi') = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi) = \mathcal{C}. \tag{2.38}$$

Finally, we note that $\pi \nparallel \mathbf{v} \Rightarrow \pi' \nparallel \mathbf{v}$, because $\pi' \parallel \mathbf{v} \Rightarrow \pi' = \pi$, hence $\pi \parallel \mathbf{v}$ (but we can also observe, more directly, that (2.38) $\Rightarrow \pi' \nparallel \mathbf{v}$).

2) Now, suppose the conic $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$ is an ellipse or a hyperbola (i.e., it is not a pair of parallel lines) tangent to $\mathscr{C}_{\mathbf{v}}$. Let OP_1, OP_2 be conjugate semi-diameters of $\mathcal{C}$.[17]

Having already observed that $\pi, \pi' \nparallel \mathbf{v}$, the thesis follows from (2.38) and from the fact that the restrictions

$$\Pi_{\mathbf{v}}\Big|_{\pi} : \pi \longrightarrow \omega \quad \text{and} \quad \Pi_{\mathbf{v}}\Big|_{\pi'} : \pi' \longrightarrow \omega \quad \text{are affine transformations.} \tag{2.39}$$

More precisely, by (2.38) and (2.39), we can certainly say that there are $Q_1, Q_2 \in \mathscr{H} \cap \pi$ and $\tilde{Q}_1, \tilde{Q}_2 \in \mathscr{H} \cap \pi'$ such that

$$\Pi_{\mathbf{v}}(Q_1) = \Pi_{\mathbf{v}}(\tilde{Q}_1) = P_1, \quad \Pi_{\mathbf{v}}(Q_2) = \Pi_{\mathbf{v}}(\tilde{Q}_2) = P_2.$$

Thus the pairs (OQ_1, OQ_2) and $(O\tilde{Q}_1, O\tilde{Q}_2)$ are conjugate semi-diameters of the conics $\mathscr{H} \cap \pi$ and $\mathscr{H} \cap \pi'$, respectively. On the other hand, it easy to show that Q_i and $\tilde{Q}_i$ are necessarily $\pi_{\mathbf{v}}-$symmetric, that is, $\tilde{Q}_i = Q_i'$ and

$$\Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathscr{H} = \{Q_i, \tilde{Q}_i\} \quad \text{for } i = 1, 2.$$

Indeed, if $Q_i \neq \tilde{Q}_i$ there is nothing to prove, because $\Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathscr{H}$ contains at most two points which are $\pi_{\mathbf{v}}-$symmetric. Conversely, if $Q_i = \tilde{Q}_i$ then $Q_i \in \pi \cap \pi'$ and this implies $Q_i \in \pi_{\mathbf{v}}$, i.e., $Q_i = Q_i'$.[21] But, if $Q_i \in \pi_{\mathbf{v}}$, the set $\Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathscr{H}$ consists of only one element because the line $\Pi_{\mathbf{v}}^{-1}(P_i)$ is then tangent to $\mathscr{H}$ at Q_i. See Claim 2.1 and Rem. 2.2.

3) Conversely, let π be a plane through the origin O such that $\pi \nparallel \mathbf{v}$. By Claim 2.5, $\mathscr{H} \cap \pi$ is an admissible conic in π and since we suppose $\pi \nparallel \mathbf{v}$, it is clear that (2.36) holds.
Thus $\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi)$ is an admissible conic in ω. Moreover, $\mathcal{C} \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$ because we assume $\mathscr{H} \cap \pi_{\mathbf{v}} \cap \pi \neq \emptyset$. Taking into account that $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$, from Cor. 2.13 we then deduce that $\mathcal{C}$ and $\mathscr{C}_{\mathbf{v}}$ are tangent at any point of $\Pi_{\mathbf{v}}\big(\mathscr{H} \cap \pi_{\mathbf{v}} \cap \pi\big)$. □

Remark 2.20 *The condition $\mathscr{H} \cap \pi_{\mathbf{v}} \cap \pi \neq \emptyset$, which appears in 3) of Claim 2.19, is certainly true if at least one of the conics $\mathscr{H} \cap \pi_{\mathbf{v}}$ and $\mathscr{H} \cap \pi$ is an ellipse. For instance, let $\mathcal{E} = \mathscr{H} \cap \pi_{\mathbf{v}}$ be an ellipse. Then*

$$\mathscr{H} \cap \pi_{\mathbf{v}} \cap \pi = \mathcal{E} \cap (\pi_{\mathbf{v}} \cap \pi) \neq \emptyset, \tag{2.40}$$

because $\mathcal{E}$ is an ellipse in $\pi_{\mathbf{v}}$ centered at O and $\pi_{\mathbf{v}} \cap \pi$ contains a line through O in $\pi_{\mathbf{v}}$.

[20] The equation of a conic $\mathcal{Q} \subset \{z = 0\}$ that is symmetrical with respect to O, but which does not pass through O, can be expressed in the form $\alpha x^2 + \beta y^2 + \gamma xy = 1$. The coefficients α, β, γ are uniquely determined if (for instance) we know two points $P_1, P_2 \in \mathcal{Q}$ and the tangent t at one of them, provided $OP_1 \nparallel OP_2$ and $O \notin t$ (if $O \in t$ then $\mathcal{Q}$ does not exist). If $P_1P_2 \parallel t$ or $P_1P_2' \parallel t$ (with P_2' symmetric to P_2 with respect to O), the conic is degenerate. Namely, in this case $\mathcal{Q} = t \cup t'$, with t' the symmetric of t with respect to O.

[21] Since π, π' are $\pi_{\mathbf{v}}-$symmetric planes,

$$Q \in \pi \cap \pi' \text{ and } Q \notin \pi_{\mathbf{v}} \quad \Longrightarrow \quad Q \neq Q' \text{ and then } \mathbf{v} \parallel QQ' \parallel \pi,$$

which is a contradiction, because we know that $\pi, \pi' \nparallel \mathbf{v}$.

2.4 The case of degenerate ellipses

In Claim 2.19 we have assumed that $\mathcal{C} \subset \Pi_{\mathbf{v}}(\mathscr{H})$ is an admissible conic, tangent to $\mathscr{C}_{\mathbf{v}}$. But in view of the proof of Thm. 1.11 we need also to consider what happen if $\mathcal{C}$ is a degenerate ellipse (in the sense of Def. 1.9) inscribed in $\mathscr{C}_{\mathbf{v}}$, when $\mathscr{C}_{\mathbf{v}}$ is a hyperbola.

Claim 2.21 *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate and let $\ell \subset \omega$ be a line through the origin O such that $\ell \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$. Let ζ be the plane through ℓ and parallel to $\mathbf{v}$. Then $\mathscr{H} \cap \zeta$ is an ellipse (hyperbola) iff $\mathscr{C}_{\mathbf{v}}$ is a hyperbola (ellipse).*

Proof. As in the proof of Claim 2.15, it suffices to prove the result for

$$\mathbf{v} = \lambda \mathbf{j} + n\mathbf{k} \quad \text{with} \quad \lambda^2 - n^2 \neq 0, \tag{2.41}$$

because $\mathscr{H}$ is invariant under rotations about the z-axis. Therefore, taking into account formula (2.27), $\mathscr{C}_{\mathbf{v}} \subset \omega$ has equation

$$x^2 + \left(\frac{n^2}{n^2 - \lambda^2}\right) y^2 = \rho^2 \quad \text{with } \rho > 0. \tag{2.42}$$

Now, by hypothesis, there exits a point $L = L(x_L, y_L, 0) \in \ell \cap \mathscr{C}_{\mathbf{v}}$. By (2.42) the coordinates of L must then satisfy the relation

$$n^2\big(x_L^2 + y_L^2\big) - \lambda^2 x_L^2 = \big(n^2 - \lambda^2\big)\rho^2. \tag{2.43}$$

On the other hand, ζ is the plane through OL and parallel to $\mathbf{v}$. Thus ζ has equation

$$\zeta : \, (ny_L)x - (nx_L)y + (\lambda x_L)z = 0. \tag{2.44}$$

Noting (2.43), by Claim 2.5, we deduce that:

- $\mathscr{H} \cap \zeta$ is an ellipse $\Leftrightarrow$ $n^2 - \lambda^2 < 0$ $\Leftrightarrow$ $\mathscr{C}_{\mathbf{v}}$ is a hyperbola;
- $\mathscr{H} \cap \zeta$ is a hyperbola $\Leftrightarrow$ $n^2 - \lambda^2 > 0$ $\Leftrightarrow$ $\mathscr{C}_{\mathbf{v}}$ is an ellipse. □

Claim 2.22 *Given $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$, let ζ be a plane through O and parallel to $\mathbf{v}$. Let $\mathcal{E} \subset \zeta$ be an ellipse with center O and let $\Pi_{\mathbf{v}}(\mathcal{E}) = MN$, for suitable $M, N \in \omega \cap \zeta$.*

1) Let (OQ_1, OQ_2) be a pair of conjugate semi-diameters for $\mathcal{E}$. If $P_1 = \Pi_{\mathbf{v}}(Q_1)$ and $P_2 = \Pi_{\mathbf{v}}(Q_2)$, then we have

$$|MN|^2 = 4\big(|OP_1|^2 + |OP_2|^2\big). \tag{2.45}$$

2) If $P_1, P_2 \in \omega \cap \zeta$ satisfy (2.45), then there are $Q_1, \widehat{Q}_1, Q_2, \widehat{Q}_2 \in \mathcal{E}$ such that $\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathcal{E} = \big\{Q_1, \widehat{Q}_1\big\}$, $\Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathcal{E} = \big\{Q_2, \widehat{Q}_2\big\}$ and (OQ_1, OQ_2), $(O\widehat{Q}_1, O\widehat{Q}_2)$ are distinct pairs of conjugate semi-diameters for $\mathcal{E}$.

Proof. 1) To begin with, we introduce orthogonal coordinates h, k in the plane ζ such that $O = (0, 0)$ and

$$\mathcal{E} : \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1 \quad \text{with} \quad a, b > 0. \tag{2.46}$$

In this situation it is well known that OQ_1, OQ_2 are conjugate semi-diameters for $\mathcal{E}$ if and only if there is $\theta \in [0, 2\pi)$ such that

$$Q_1 = \big(a\cos\theta, b\sin\theta\big) \quad \text{and} \quad Q_2 = \pm\big(a\sin\theta, -b\cos\theta\big).^{22} \tag{2.47}$$

Moreover, since $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is linear, given a unit vector $\mathbf{u}$ such that $\mathbf{u} \parallel \omega \cap \zeta$, there are $\alpha, \beta \in \mathbb{R}$ (not both zero) such that

$$\overrightarrow{OP_1} = \big(a\alpha\cos\theta + b\beta\sin\theta\big)\mathbf{u} \quad \text{and} \quad \overrightarrow{OP_2} = \pm\big(a\alpha\sin\theta - b\beta\cos\theta\big)\mathbf{u}, \tag{2.48}$$

for all $\theta \in [0, 2\pi)$. From (2.48) we immediately have

$$|OP_1|^2 + |OP_2|^2 = (a\alpha)^2 + (b\beta)^2 \quad \text{for all } \theta \in [0, 2\pi) \tag{2.49}$$

and, in particular,

$$|OM|^2 = |ON|^2 = (a\alpha)^2 + (b\beta)^2, \tag{2.50}$$

because $\Pi(Q_1) = M$ or N when $\Pi(Q_2) = O$, that is, when OQ_2 is parallel to the projection direction $\mathbf{v}$. Hence, from (2.50) we deduce that $|MN|^2 = 4(a\alpha)^2 + 4(b\beta)^2$, because $O = \frac{M+N}{2}$.

2) Conversely, let $P_1, P_2 \in \omega \cap \zeta$ such that the relation (2.45) is true. Before proceeding, let's remember that the ellipse $\mathcal{E}$ has oblique symmetry, in the direction of $\mathbf{v}$, with respect to the line, say $l_{\mathbf{v}}$, through O and parallel to the direction conjugate to that of $\mathbf{v}$. Thus, if

$$\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathcal{E} = \{R_1, \widehat{R}_1\} \quad \text{and} \quad \Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathcal{E} = \{R_2, \widehat{R}_2\},$$

it is clear that the points R_1 and R_2 are obliquely symmetrical (in the direction of $\mathbf{v}$ and with respect to $l_{\mathbf{v}}$) to $\widehat{R}_1$ and $\widehat{R}_2$, respectively. In addition, we know that $R_1 = \widehat{R}_1 \Leftrightarrow R_1 \in l_{\mathbf{v}} \Leftrightarrow P_1 = M$ or N (i.e., $P_2 = O$, by (2.45)) and, similarly, for the couple $R_2, \widehat{R}_2$.

Now, starting for instance from R_1, we certainly have

$$R_1 = (a\cos\theta_1, b\sin\theta_1) \quad \text{for a suitable } \theta_1 \in [0, 2\pi). \tag{2.51}$$

By (2.45) and taking into account (2.48) and (2.49), one of the following must hold:

$$P_2 = \Pi_{\mathbf{v}}\big(a\sin\theta_1, -b\cos\theta_1\big) \quad \text{or} \quad P_2 = \Pi_{\mathbf{v}}\big(-a\sin\theta_1, b\cos\theta_1\big), \tag{2.52}$$

because in $\omega \cap \zeta$ there are only two points at a distance of $\frac{1}{2}\sqrt{|MN|^2 - 4|OP_1|^2}$ from O. Assuming, for example, that the second of (2.52) holds, we define

$$Q_1 = R_1 = (a\cos\theta_1, b\sin\theta_1) \quad \text{and} \quad Q_2 = (-a\sin\theta_1, b\cos\theta_1).^{23} \tag{2.53}$$

Then, by the condition (2.47) above, (OQ_1, OQ_2) is a pair of conjugate semi-diameters such that $\Pi_{\mathbf{v}}(Q_1) = P_1$ and $\Pi_{\mathbf{v}}(Q_2) = P_2$. Finally, denoting with $\widehat{Q}_1$ and $\widehat{Q}_2$ the symmetric to Q_1 and Q_2 respectively, we can easily see that

$$(O\widehat{Q}_1, O\widehat{Q}_2) \tag{2.54}$$

[22] See [10], p. 39.

[23] Clearly, we have $Q_2 = R_2$ or $\widehat{R}_2$.

gives a pair of conjugate semi-diameters such that $(O\widehat{Q}_1, O\widehat{Q}_2) \neq (OQ_1, OQ_2)$. Indeed, let us suppose, for instance, that $\widehat{Q}_1 = Q_1$. Then, as we observed above, $P_1 = M$ or N and $P_2 = O$. But, in turn, the condition $P_2 = O$ implies $\widehat{Q}_2 \neq Q_2$. □

To conclude, we assume that $OP_1, OP_2 \subset \omega$ do not both vanish and that $OP_1 \parallel OP_2$. Then we consider the degenerate ellipse $\mathcal{E}_{P_1,P_2} = MN$, according to Def. 1.9. Applying Claims 2.21 and 2.22, we deduce the following:

Claim 2.23 *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be non-degenerate and such that $\mathscr{C}_{\mathbf{v}}$ is a hyperbola. Besides, let $\mathcal{E}_{P_1,P_2} = MN$ be a degenerate ellipse inscribed in $\mathscr{C}_{\mathbf{v}}$ and let ζ be the plane through MN and parallel to the projection direction given by $\mathbf{v}$.*

Then $\mathscr{H} \cap \zeta$ is an ellipse, with center O, such that $\Pi_{\mathbf{v}}(\mathscr{H} \cap \zeta) = \mathcal{E}_{P_1,P_2}$. Furthermore, there are $Q_1, Q_1', Q_2, Q_2' \in \mathscr{H} \cap \zeta$ such that $\Pi_{\mathbf{v}}^{-1}(P_1) \cap \mathscr{H} = \{Q_1, Q_1'\}$, $\Pi_{\mathbf{v}}^{-1}(P_2) \cap \mathscr{H} = \{Q_2, Q_2'\}$ and (OQ_1, OQ_2), (OQ_1', OQ_2') are distinct pairs of conjugate semi-diameters of $\mathscr{H} \cap \zeta$.

Proof. Since MN is a segment through the origin O and $M, N \in \mathscr{C}_{\mathbf{v}}$, by Claim 2.21 we know that $\mathcal{E} = \mathscr{H} \cap \zeta$ is an ellipse, with center O. Then we can easily see that

$$\Pi_{\mathbf{v}}(\mathcal{E}) = \mathcal{E}_{P_1,P_2}. \tag{2.55}$$

Indeed, assuming $\mathcal{E}_{P_1,P_2} = MN$ inscribed in the hyperbola $\mathscr{C}_{\mathbf{v}}$, we have: $\mathcal{E}_{P_1,P_2} \subset \Pi_{\mathbf{v}}(\mathcal{E})$, because $\mathcal{E}_{P_1,P_2} \subset \mathbf{int}(\mathscr{C}_{\mathbf{v}})$, and also $\mathcal{E}_{P_1,P_2} \supset \Pi_{\mathbf{v}}(\mathcal{E})$ because $M, N \in \mathscr{C}_{\mathbf{v}}$. To proceed, we recall that $\mathcal{E}_{P_1,P_2} = MN$ implies

$$|MN|^2 = 4\big(|OP_1|^2 + |OP_2|^2\big). \tag{2.56}$$

Moreover, we note that

$$\Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathscr{H} = \Pi_{\mathbf{v}}^{-1}(P_i) \cap \mathcal{E} \quad \text{for } i = 1, 2$$

and that $\mathcal{E}$ has oblique symmetry, in the direction of $\mathbf{v}$, with respect to the line $l_{\mathbf{v}} = \zeta \cap \pi_{\mathbf{v}}$.[24] We can therefore apply part 2) of Claim 2.22 with $\mathcal{E} = \mathscr{H} \cap \zeta$ and $l_{\mathbf{v}} = \zeta \cap \pi_{\mathbf{v}}$ and this immediately gives the thesis. □

2.5 Some properties of the tangent planes of $\mathscr{H}$

Definition 2.24 *Given $P \in \mathscr{H}$, we denote with $T_{\mathscr{H}}(P)$ the tangent plane to $\mathscr{H}$ at P.*

If $P \in \mathscr{H}$, $P = P(x_P, y_P, z_P)$, we recall that

$$T_{\mathscr{H}}(P): \ x_P x + y_P y - z_P z = \rho^2. \tag{2.57}$$

Claim 2.25 *If $P, Q \in \mathscr{H}$ and O is the origin of coordinates, then*

$$OP \parallel T_{\mathscr{H}}(Q) \quad \Leftrightarrow \quad OQ \parallel T_{\mathscr{H}}(P). \tag{2.58}$$

Proof. Indeed, given $P = P(x_P, y_P, z_P) \in \mathscr{H}$ and $Q = Q(x_Q, y_Q, z_Q)$, we have that

$$OQ \parallel T_{\mathscr{H}}(P) \quad \Leftrightarrow \quad x_P x_Q + y_P y_Q - z_P z_Q = 0. \tag{2.59}$$

But the last condition of (2.59) is symmetric with respect to P and Q if $P, Q \in \mathscr{H}$. □

Taking into account the oblique symmetry of $\mathscr{H}$ with respect to the plane $\pi_{\mathbf{v}}$ (Def. 1.2), applying Claim 2.25 we easily get the following:

[24] It turns out that $l_{\mathbf{v}} = \zeta \cap \pi_{\mathbf{v}}$ is the line, through O, parallel to the direction conjugate to that of $\mathbf{v}$.

Corollary 2.26 *If $P, Q \in \mathscr{H}$ and P', Q' are $\pi_{\mathbf{v}}-$symmetric to P, Q respectively, then*

$$OP \parallel T_{\mathscr{H}}(Q) \quad \Leftrightarrow \quad OP' \parallel T_{\mathscr{H}}(Q') \tag{2.60}$$

and

$$OP \parallel T_{\mathscr{H}}(Q') \quad \Leftrightarrow \quad OQ \parallel T_{\mathscr{H}}(P'). \tag{2.61}$$

Proof. Recalling Def. 2.3 and Rem. 2.4, we easily have

$$\mathtt{S}_{\mathbf{v}}\big(T_{\mathscr{H}}(Q)\big) = T_{\mathscr{H}}(Q'), \tag{2.62}$$

where $\mathtt{S}_{\mathbf{v}}$ is the oblique symmetry with respect to the plane $\pi_{\mathbf{v}}$, in the direction of $\mathbf{v}$. This immediately gives (2.60). Then (2.61) follows from (2.60) and Claim 2.25. □

Definition 2.27 *Assuming $OP \nparallel OQ$, we denote with $\langle O, P, Q\rangle$ the plane through the origin O and the points P, Q. With $\mathscr{C}(P, Q)$ we indicate the admissible conic*

$$\mathscr{C}(P, Q) \stackrel{\text{def}}{=} \mathscr{H} \cap \langle O, P, Q\rangle. \tag{2.63}$$

Moreover, given $R \in \mathscr{C}(P, Q)$, we will denote with $T_{\mathscr{C}(P,Q)}(R) \subset \langle O, P, Q\rangle$ the tangent line to $\mathscr{C}(P, Q)$ passing through the point R.

Remark 2.28 *By* (2.57) *and* (2.59)*, if $P \in \mathscr{H}$ then $OP \nparallel T_{\mathscr{H}}(P)$. More generally,*

$$P, Q \in \mathscr{H} \;\text{ and }\; OQ \parallel T_{\mathscr{H}}(P) \quad \Rightarrow \quad OP \nparallel OQ, \tag{2.64}$$

because $OP \parallel OQ \;\Rightarrow\; OP \parallel T_{\mathscr{H}}(P)$. Further, if $OP \nparallel OQ$ and $R \in \mathscr{H}$ then

$$T_{\mathscr{H}}(R) \cap \langle O, P, Q\rangle \neq \emptyset \quad \Rightarrow \quad T_{\mathscr{H}}(R) \nparallel \langle O, P, Q\rangle, \tag{2.65}$$

because, by (2.57), $O \notin T_{\mathscr{H}}(R)$. In particular, this implies that the plane $\langle O, P, Q\rangle$ has always transverse intersection (i.e., it is never tangent) with the hyperboloid $\mathscr{H}$.

Claim 2.29 *Suppose $P, Q \in \mathscr{H}$. Then $OP \parallel T_{\mathscr{H}}(Q) \Leftrightarrow OP \nparallel OQ$ and $\mathscr{C}(P, Q) = \mathscr{H} \cap \langle O, P, Q\rangle$ is an ellipse with (OP, OQ) as a pair of conjugate semi-diameters.*

Proof. $\Rightarrow$ By the first part of Rem. 2.28, we already know that $OP \nparallel OQ$. This implies that $\mathscr{C}(P, Q) = \mathscr{H} \cap \langle O, P, Q\rangle$ is an admissible conic in the sense of Def. 2.6. In particular, $\mathscr{C}(P, Q)$ admits tangent line in each of its points. Besides, by the second part of Rem. 2.28, $T_{\mathscr{H}}(Q) \nparallel \langle O, P, Q\rangle$. Hence we deduce that the tangent line $T_{\mathscr{C}(P,Q)}(Q)$ satisfies

$$T_{\mathscr{C}(P,Q)}(Q) = T_{\mathscr{H}}(Q) \,\cap \langle O, P, Q\rangle, \tag{2.66}$$

because it is clear that $T_{\mathscr{C}(P,Q)}(Q) \subset T_{\mathscr{H}}(Q)$ and that $T_{\mathscr{C}(P,Q)}(Q) \subset \langle O, P, Q\rangle$.
Then, since $OP \parallel \langle O, P, Q\rangle$ and we suppose $OP \parallel T_{\mathscr{H}}(Q)$, it follows that

$$OP \parallel T_{\mathscr{C}(P,Q)}(Q). \tag{2.67}$$

Moreover, by Claim 2.25, $OP \parallel T_{\mathscr{H}}(Q) \;\Leftrightarrow\; OQ \parallel T_{\mathscr{H}}(P)$. So with the same arguments used above we can prove that

$$OQ \parallel T_{\mathscr{C}(P,Q)}(P). \tag{2.68}$$

From this we deduce that $\mathscr{C}(P,Q)$ must be an ellipse, because (2.67) and (2.68) cannot both be true if $\mathscr{C}(P,Q)$ is a hyperbola or a pair of distinct, parallel lines which are symmetric with respect to the origin O. [25] Having proved that $\mathscr{C}(P,Q)$ is an ellipse, again from (2.67) and (2.68), we deduce that (OP, OQ) is a pair of conjugate semi-diameters.

$\Leftarrow$ The inverse implication is immediate from the properties of semi-diameters of an ellipse. $\square$

To proceed, taking into account Defs. 1.6, 1.9, we can state the following:

Claim 2.30 *Let $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ be a parallel projection. Let $Q_1, Q_2 \in \mathscr{H}$ such that $OQ_1 \,\|\, T_{\mathscr{H}}(Q_2)$ and let $P_1 = \Pi_{\mathbf{v}}(Q_1)$, $P_2 = \Pi_{\mathbf{v}}(Q_2)$. Then we have:*

(1) *If $OP_1 \nparallel OP_2$, then $\Pi_{\mathbf{v}}\Big|_{\langle O,Q_1,Q_2\rangle} : \langle O, Q_1, Q_2\rangle \to \omega$ defines an affine map such that*

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_1,Q_2)) = \mathcal{E}_{P_1,P_2}. \tag{2.69}$$

If we further suppose that $\Pi_{\mathbf{v}}$ is non-degenerate, then $\mathcal{E}_{P_1,P_2}$ is tangent to $\mathscr{C}_{\mathbf{v}}$.

(2) *If $OP_1 \,\|\, OP_2$, then $\Pi_{\mathbf{v}}(\mathscr{C}(Q_1,Q_2))$ is the degenerate ellipse $\mathcal{E}_{P_1,P_2}$ determined by the segments OP_1, OP_2. If we further assume that $\Pi_{\mathbf{v}}$ is non-degenerate, then $\mathscr{C}_{\mathbf{v}}$ is necessarily a hyperbola and $\mathscr{C}_{\mathbf{v}}$ circumscribes $\mathcal{E}_{P_1,P_2}$ (in the sense Def. 1.9).*

Proof. By Claim 2.29, we already know that $OQ_1 \nparallel OQ_2$ and that $\mathscr{C}(Q_1,Q_2)$ is an ellipse with conjugate semi-diameters OQ_1, OQ_2. Besides, having $\Pi_{\mathbf{v}}(Q_1) = P_1$, $\Pi_{\mathbf{v}}(Q_2) = P_2$ with $OQ_1 \nparallel OQ_2$, the segments OP_1, OP_2 cannot both vanish. Hence we may consider the (eventually degenerate) ellipse $\mathcal{E}_{P_1,P_2}$.

(1) In this case, we have that

$$OP_1 \nparallel OP_2 \quad \text{and} \quad \Pi_{\mathbf{v}}(Q_1) = P_1,\ \Pi_{\mathbf{v}}(Q_2) = P_2 \implies \mathbf{v} \nparallel \langle O, Q_1, Q_2\rangle. \tag{2.70}$$

So the restriction

$$\Pi_{\mathbf{v}}\Big|_{\langle O,Q_1,Q_2\rangle} : \langle O, Q_1, Q_2\rangle \to \omega$$

defines an affine transformation. Having $\Pi_{\mathbf{v}}(OQ_1) = OP_1$ and $\Pi_{\mathbf{v}}(OQ_2) = OP_2$, it is therefore clear that (2.69) holds. Next, we define

$$l = \langle O, Q_1, Q_2\rangle \cap \pi_{\mathbf{v}}.$$

Noting that l is a straight line through the origin O in $\langle O, Q_1, Q_2\rangle$, or all plane $\langle O, Q_1, Q_2\rangle$, it is clear that

$$\mathscr{C}(Q_1,Q_2) \cap l \neq \emptyset, \tag{2.71}$$

because $\mathscr{C}(Q_1,Q_2) = \mathscr{H} \cap \langle O, Q_1, Q_2\rangle$ is an ellipse, centered at O, in $\langle O, Q_1, Q_2\rangle$. Hence

$$\mathscr{C}(Q_1,Q_2) \cap (\mathscr{H} \cap \pi_{\mathbf{v}}) = \mathscr{H} \cap \langle O, Q_1, Q_2\rangle \cap \pi_{\mathbf{v}} = \mathscr{C}(Q_1,Q_2) \cap l \neq \emptyset. \tag{2.72}$$

[25] If $\mathscr{C}(P,Q)$ is an hyperbola, just note what happens for $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Given $Q = (x_q, y_q) \in \mathcal{H}$ and $P = (x_p, y_p)$, it follows that $OP \parallel T_{\mathcal{H}}(Q)$ iff $\frac{x_q x_p}{a^2} - \frac{y_q y_p}{b^2} = 0$. This means that $x_p = k\frac{y_q}{b^2}$, $y_p = k\frac{x_q}{a^2}$ for some $k \in \mathbb{R}$. But then $\frac{x_p^2}{a^2} - \frac{y_p^2}{b^2} = \frac{k^2}{a^2b^2}(\frac{y_q^2}{b^2} - \frac{x_q^2}{a^2}) = -\frac{k^2}{a^2b^2}$. Thus $P \notin \mathcal{H}$ regardless of the value of k. If $\mathscr{C}(P,Q)$ is a pair of distinct, parallel lines which are symmetric with respect to O, it is obvious that (2.67), (2.68) cannot hold.

This, in turn, implies that

$$\mathcal{E}_{P_1,P_2} \cap \Pi_{\mathbf{v}}\big(\mathscr{H} \cap \pi_{\mathbf{v}}\big) = \Pi_{\mathbf{v}}(\mathscr{C}(Q_1, Q_2)) \cap \Pi_{\mathbf{v}}\big(\mathscr{H} \cap \pi_{\mathbf{v}}\big) \neq \emptyset. \tag{2.73}$$

Then, if we suppose $\Pi_{\mathbf{v}}$ is non-degenerate, (2.73) gives

$$\mathcal{E}_{P_1,P_2} \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset. \tag{2.74}$$

By 3) of Claim 2.19, $\mathcal{E}_{P_1,P_2}$ and $\mathscr{C}_{\mathbf{v}}$ are therefore tangent at any point of $\mathcal{E}_{P_1,P_2} \cap \mathscr{C}_{\mathbf{v}}$.

(2) Assuming $OP_1 \parallel OP_2$ it follows that $\zeta = \langle O, Q_1, Q_2 \rangle$ is the plane through O, P_1, P_2 and parallel to the vector $\mathbf{v}$. We can then apply part 1) of Claim 2.22 with $\mathcal{E} = \mathscr{C}(Q_1, Q_2)$. It easily follows that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_1, Q_2)) = MN = \mathcal{E}_{P_1,P_2}, \tag{2.75}$$

because we already know that OQ_1, OQ_2 are conjugate semi-diameters of $\mathscr{C}(Q_1, Q_2)$ and, by (2.45), we have $|MN|^2 = 4\big(|OP_1|^2 + |OP_2|^2\big)$.

To proceed, since $\mathscr{C}(Q_1, Q_2)$ is an ellipse in $\zeta = \langle O, Q_1, Q_2 \rangle$, we can prove as in case (1) above that $\mathcal{E}_{P_1,P_2} \cap \Pi_{\mathbf{v}}\big(\mathscr{H} \cap \pi_{\mathbf{v}}\big) \neq \emptyset$. If we now suppose $\Pi_{\mathbf{v}}$ is non-degenerate, we have

$$MN \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset. \tag{2.76}$$

This means that the line ℓ through M, N is a line through O such that $\ell \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$. Then, applying Claim 2.21, we see that $\mathscr{C}_{\mathbf{v}}$ must be a hyperbola, because $\mathscr{C}(Q_1, Q_2) = \mathscr{H} \cap \zeta$ is an ellipse. [26] Finally, $\mathscr{C}_{\mathbf{v}}$ circumscribes $\mathcal{E}_{P_1,P_2}$. In fact, we have shown above that $MN \cap \mathscr{C}_{\mathbf{v}} \neq \emptyset$ and, by Cor. 2.13, we know that $MN \subset \mathbf{int}(\mathscr{C}_{\mathbf{v}})$. So we have $M, N \in \mathscr{C}_{\mathbf{v}}$, since M, N (as well $\mathscr{C}_{\mathbf{v}}$) are symmetrical with respect to the origin O. □

Remark 2.31 *Under the assumptions of (1) of Claim 2.30 and taking into account Defs. 2.7, 2.8 and Cor. 2.13, if the projection $\Pi_{\mathbf{v}}$ is non-degenerate we can also say that:*

- *$\mathscr{C}_{\mathbf{v}}$ is inscribed in $\mathcal{E}_{P_1,P_2}$, if $\mathscr{C}_{\mathbf{v}}$ is an ellipse. In particular, we have $\mathscr{C}_{\mathbf{v}} = \mathcal{E}_{P_1,P_2}$ if and only if $\pi_{\mathbf{v}} = \langle O, Q_1, Q_2 \rangle$.*
- *$\mathscr{C}_{\mathbf{v}}$ circumscribes $\mathcal{E}_{P_1,P_2}$ if $\mathscr{C}_{\mathbf{v}}$ is a hyperbola.*

3 Hyperbolic Pohlke's projection in the circular case

In this section we will explicitly determine the hyperbolic Pohlke's projection $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ when in Def. 1.4 we also assume that two of the segments OP_1, OP_2, OP_3 are equal and perpendicular. Before proceeding we recall that, according to Def. 1.2, the points P, P' are $\pi_{\mathbf{v}}$–symmetric if P, P' are obliquely symmetrical with respect to the plane $\pi_{\mathbf{v}}$, in the direction of $\mathbf{v}$. That is, $P' = \mathtt{S}_{\mathbf{v}}(P)$ where $\mathtt{S}_{\mathbf{v}}$ is the map introduced in Def. 2.3. Moreover, if $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ is a hyperbolic Pohlke's projection in the sense of Def. 1.4, we note that:

Remark 3.1 *Considering the symmetries $\mathtt{S}_{\mathbf{v}}$, with respect to $\pi_{\mathbf{v}}$, and $\mathtt{S}_{\mathbf{k}}$, with respect to the plane $\pi_{\mathbf{k}} = \omega$ (i.e. the usual symmetry with respect to ω), it is immediate to see that:*

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[26] Noting (2.75), we may deduce directly from Cor. 2.13 that $\mathscr{C}_{\mathbf{v}}$ must be a hyperbola. In fact, we have $O \in MN \subset \Pi_{\mathbf{v}}(\mathscr{H})$ and this means that $\mathscr{C}_{\mathbf{v}}$ cannot be an ellipse.

- *If* $Q_1, Q_2, Q_3 \in \mathscr{H}$ *satisfy the conditions* (1.15), (1.16) *of Def. 1.4 then, by Cor. 2.26, also the points* $Q_1' = \mathtt{S}_{\mathbf{v}}(Q_1)$, $Q_2' = \mathtt{S}_{\mathbf{v}}(Q_2)$ $Q_3' = \mathtt{S}_{\mathbf{v}}(Q_3)$ *satisfy* (1.15), (1.16). *This means that in Def. 1.4 the triads* Q_1, Q_2, Q_3 *and* Q_1', Q_2', Q_3' *are perfectly equivalent.*

- *Let us denote with* $\bar{\Pi}_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ *the symmetric projection with respect to* ω*, i.e.,*

$$\bar{\Pi}_{\mathbf{v}}(P) = \Pi_{\mathbf{v}}(\mathtt{S}_{\mathbf{k}}(P)) \quad \textit{for } P \in \mathbb{R}^3. \tag{3.1}$$

Then $\bar{\Pi}_{\mathbf{v}}$*, with the points* $\mathtt{S}_{\mathbf{k}}(Q_1)$, $\mathtt{S}_{\mathbf{k}}(Q_2)$ *and* $\mathtt{S}_{\mathbf{k}}(Q_3)$*, still gives a hyperbolic Pohlke's projection for* OP_1, OP_2, OP_3*. Observe also that if* $\bar{\mathbf{v}} = \mathtt{S}_{\mathbf{k}}(\mathbf{v})$*, then*

$$\bar{\Pi}_{\mathbf{v}} = \Pi_{\bar{\mathbf{v}}} \quad \textit{and} \quad \Pi_{\bar{\mathbf{v}}}(\mathscr{H} \cap \pi_{\bar{\mathbf{v}}}) = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}). \tag{3.2}$$

3.1 The circular case

We consider here the problem of determining the hyperbolic Pohlke's projections $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ in the *circular case*. More precisely, for $OP_1, OP_2, OP_3 \subset \omega$ such that

$$OP_1 \perp OP_2 \quad \text{and} \quad |OP_1| = |OP_2| = 1. \tag{3.3}$$

To begin with, according to Def. 1.4, we need to find $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ non-degenerate and then $Q_1, Q_2 \in \mathscr{H}(\rho)$ such that

$$\Pi_{\mathbf{v}}(Q_1) = P_1, \;\; \Pi_{\mathbf{v}}(Q_2) = P_2 \;\; \text{with} \;\; OQ_1 \parallel T_{\mathscr{H}}(Q_2).$$

Assuming such a projection exists, from (1) of Claim 2.30 we deduce that $\mathcal{E}_{P_1,P_2}$ must be tangent to $\mathscr{C}_{\mathbf{v}}$. Since $\mathscr{E}_{P_1,P_2}$ is the circle with center O and radius $r = 1$, we have two possibilities:

- If $\mathscr{C}_{\mathbf{v}}$ is an ellipse (circle), having to be inscribed in $\mathscr{E}_{P_1,P_2}$ (by (1) of Cor. 2.13), $\mathscr{C}_{\mathbf{v}}$ must have semi-major axis $a = 1$ (radius $r = 1$).

- If $\mathscr{C}_{\mathbf{v}}$ is a hyperbola, having to circumscribe $\mathscr{E}_{P_1,P_2}$ (by (2) of Cor. 2.13), $\mathscr{C}_{\mathbf{v}}$ must have transverse semi-axis $a = 1$.

Then, from Claim 2.15 and Rem. 2.17, we conclude that:

Claim 3.2 *If* (3.3) *holds and if there is a hyperbolic Pohlke's projection for* OP_1, OP_2, OP_3 *(according to Def. 1.4), then* $\rho = 1$*. That is, we have*

$$\mathscr{H} = \mathscr{H}(1) = \big\{(x, y, z) \in \mathbb{R}^3 \,\big|\, x^2 + y^2 - z^2 = 1\big\}. \tag{3.4}$$

After this, again assuming that the hyperbolic Pohlke's projection $\Pi_{\mathbf{v}}$ exists, we note that (3.3), (3.4) imply $P_1, P_2 \in \mathscr{H}$. Thus we must have:

$$P_1 = Q_1 \text{ or } Q_1' \quad \text{and} \quad P_2 = Q_2 \text{ or } Q_2'.\,^{27} \tag{3.5}$$

But to satisfy the conditions of Def. 1.4 it is necessary to set

$$Q_1 = P_1 \;\; \text{and} \;\; Q_2 = P_2 \tag{3.6}$$

[27] Given $Q \in \mathscr{H}$, by (2.59) we have $OP \parallel T_{\mathscr{H}}(Q) \Leftrightarrow x_P x_Q + y_P y_Q - z_P z_Q = 0$. Therefore, if $P_1, P_2 \in \omega$ are such that $OP_1 \perp OP_2$, then $OP_1 \parallel T_{\mathscr{H}}(P_2)$ and $OP_2 \parallel T_{\mathscr{H}}(P_1)$.

or, equivalently, $Q_1' = P_1$ and $Q_2' = P_2$.[28] In fact, if we set $Q_1 = P_1$ and $Q_2' = P_2$, applying Cor. 2.26, we find:

$$OQ_3 \parallel T_{\mathscr{H}}(Q_1') \;\Leftrightarrow\; OQ_1 \parallel T_{\mathscr{H}}(Q_3') \;\Leftrightarrow\; OP_1 \parallel T_{\mathscr{H}}(Q_3'), \tag{3.7}$$

$$OQ_2 \parallel T_{\mathscr{H}}(Q_3) \;\Leftrightarrow\; OQ_2' \parallel T_{\mathscr{H}}(Q_3') \;\Leftrightarrow\; OP_2 \parallel T_{\mathscr{H}}(Q_3'). \tag{3.8}$$

Now, from (2.59), it is easy to see that

$$OP_1 \parallel T_{\mathscr{H}}(Q_3') \text{ and } OP_2 \parallel T_{\mathscr{H}}(Q_3') \implies OQ_3' \perp \omega \,^{29} \tag{3.9}$$

and the latter condition cannot be satisfied if $Q_3' \in \mathscr{H}$. Since the same argument works if we try to define $Q_1' = P_1$ and $Q_2 = P_2$, we are forced to assume (3.6). Moreover, by choosing $Q_1 = P_1$ and $Q_2 = P_2$, we must also have

$$Q_3 \neq Q_3'. \tag{3.10}$$

Indeed, if $Q_3 = Q_3'$, from Cor. 2.26 and condition (1.16) we easily deduce that $OP_1 \parallel T_{\mathscr{H}}(Q_3)$ and $OP_2 \parallel T_{\mathscr{H}}(Q_3)$. Hence, as in (3.9), we find $OQ_3 \perp \omega$ which cannot be satisfied. In conclusion, noting that (3.10) implies $Q_3Q_3' \parallel \mathbf{v}$, we can say that:

Conditions 3.3 *Having fixed the points $Q_1 = P_1, Q_2 = P_2$ as in (3.6), to have a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3 as in (3.3), it is necessary and sufficient to determine $Q_3, Q_3' \in \mathscr{H}(1)$, $Q_3 \neq Q_3'$, such that the following conditions are true:*

(a) $OP_2 \parallel T_{\mathscr{H}}(OQ_3)$ *and* $OP_1 \parallel T_{\mathscr{H}}(OQ_3')$ $\big($*i.e.,* $OQ_3 \parallel T_{\mathscr{H}}(OP_1')$, *by Cor. 2.26*$\big)$;

(b) $Q_3Q_3' \nparallel \omega$, *because Q_3Q_3' gives the direction of projection onto ω;*

(c) Q_3, Q_3', P_3 *are collinear (i.e.,* $\Pi_{\mathbf{v}}(Q_3) = \Pi_{\mathbf{v}}(Q_3') = P_3$*);*

(d) $\mathbf{v} = \overrightarrow{Q_3Q_3'}$ *gives a non-degenerate projection direction.*

3.2 Explicit determination of $\Pi_{\mathbf{v}}$ in the circular case

To proceed, we may suppose that the coordinate axes x, y are oriented in space such that

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad P_3 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \tag{3.11}$$

In particular, in this system we have

$$\overrightarrow{OP_3} = x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}\,. \tag{3.12}$$

[28] In the following will not distinguish between these two possibilities because, by Rem. 3.1, we know that the triads Q_1, Q_2, Q_3 and Q_1', Q_2', Q_3' are equivalent.

[29] Given $Q = (x_Q, y_Q, z_Q) \in \mathscr{H}$ and $P_1, P_2 \in \omega$ such that $OP_1 \nparallel OP_2$, we have that $OP_1, OP_2 \parallel T_{\mathscr{H}}(Q) \Leftrightarrow x_Q = y_Q = 0$. But the latter condition is equivalent to $OQ \perp \omega$.

Then, taking into account (2.59), we see that (a) in Cond. 3.3 is satisfied iff $Q_3 \in \mathscr{H} \cap \{y=0\}$ and $Q_3' \in \mathscr{H} \cap \{x=0\}$. Thus we can express Q_3 and Q_3' in the form

$$Q_3 = \begin{pmatrix} \cosh^*\alpha \\ 0 \\ \sinh\alpha \end{pmatrix} \quad \text{and} \quad Q_3' = \begin{pmatrix} 0 \\ \cosh^*\beta \\ \sinh\beta \end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}), \tag{3.13}$$

where, for simplicity, we have set

$$\cosh^* t \stackrel{\text{def}}{=} \pm \cosh t.^{30} \tag{3.14}$$

Having (3.13), it is clear that $Q_3 \neq Q_3'$ and that (b) in Cond. 3.3 holds iff

$$\sinh\alpha \neq \sinh\beta. \tag{3.15}$$

Besides, (c) of Cond. 3.3 is verified iff $P_3 = Q_3 + t\,\overrightarrow{Q_3Q_3'}$ for some $t \in \mathbb{R}$. That is,

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh^*\alpha \\ 0 \\ \sinh\alpha \end{pmatrix} + t \begin{pmatrix} -\cosh^*\alpha \\ \cosh^*\beta \\ \sinh\beta - \sinh\alpha \end{pmatrix} \quad \text{for some } t \in \mathbb{R}. \tag{3.16}$$

Now, assuming that (3.15) holds, we will first study the solvability of the system (3.16) and then we will verify if also (d) of in Cond. 3.3 is satisfied, i.e., if the projection direction found is non-degenerate. We will distinguish two cases to this aim:

3.3 Case $OP_3 \parallel OP_1$ or $OP_3 \parallel OP_2$

Suppose first $OP_3 \parallel OP_2$, that is $x = 0$. Since $\cosh^*\alpha \neq 0$, the first equation of (3.16) gives $t = 1$. Then, considering also the third equation, we find $\sinh\beta = 0$. Thus $\cosh^*\beta = \pm 1$ and $\sinh\alpha \neq 0$. Summarizing up, when $x = 0$ system (3.16) is solvable iff

$$P_3 = \pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.17}$$

If (3.17) holds, then we have

$$Q_3 = \begin{pmatrix} \cosh^*\alpha \\ 0 \\ \sinh\alpha \end{pmatrix} \text{ with } \alpha \neq 0, \quad Q_3' = P_3. \tag{3.18}$$

Noting that the projection direction is given by $\mathbf{v} = \overrightarrow{Q_3Q_3'} = -(\cosh^*\alpha)\,\mathbf{i} \pm \mathbf{j} - (\sinh\alpha)\,\mathbf{k}$, condition (d) is certainly true because $(\cosh^*\alpha)^2 + 1 - \sinh^2\alpha = 2$. In conclusion, when $x = 0$ there are no hyperbolic Pohlke's projections if (3.17) fails, infinitely many if (3.17) holds. $\square$

[30] Just to have a simple parametrization of the entire hyperbolas $\mathscr{H} \cap \{y=0\}$ and $\mathscr{H} \cap \{x=0\}$, suitable for subsequent calculations.

Now suppose $OP_3 \parallel OP_1$, that is $y = 0$. Reasoning as in the previous case, we find that when $y = 0$ (3.16) is solvable iff

$$P_3 = \pm \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.19}$$

If (3.19) holds, then we have

$$Q_3 = P_3, \qquad Q_3' = \begin{pmatrix} 0 \\ \cosh^*\beta \\ \sinh\beta \end{pmatrix} \text{ with } \beta \neq 0. \tag{3.20}$$

As above, (d) of Cond. 3.3 is true because $\mathbf{v} = \overrightarrow{Q_3Q_3'} = \pm\mathbf{i} + (\cosh^*\beta)\mathbf{j} + (\sinh\beta)\,\mathbf{k}$. Thus there are no hyperbolic Pohlke's projections if (3.19) fails, infinitely many if (3.19) holds. □

Summing up, taking into account Cond.3.3, we have proved that:

Lemma 3.4 *If (3.3) is verified and $OP_3 \parallel OP_1$ (or $OP_3 \parallel OP_2$) then there are infinitely many hyperbolic Pohlke's projection for OP_1, OP_2, OP_3 if $|OP_3| = 1$, none if $|OP_3| \neq 1$.*

3.4 Case $OP_3 \nparallel OP_1$ and $OP_3 \nparallel OP_2$, that is $x, y \neq 0$

We note first that the condition $x, y \neq 0$ in (3.16) implies

$$\sinh\alpha,\ \sinh\beta \neq 0. \tag{3.21}$$

Indeed, if $\sinh\alpha = 0$, (3.15) and the third equation of (3.16) give $t = 0$. Then the second equation of (3.16) implies $y = 0$, contrary to our assumption. Similarly we find that $\sinh\beta \neq 0$.

Taking into account this fact, we deduce now a set of necessary conditions for the point P_3 to be collinear with Q_3, Q_3' (i.e., to satisfy (3.16) for some $t \in \mathbb{R}$) when (3.15) and (3.21) hold. After that, we will prove that these conditions are also sufficient.

Assuming that (3.16) is true, by (3.15) and the third equation of (3.16), we have

$$t = \frac{\sinh\alpha}{\sinh\alpha - \sinh\beta}. \tag{3.22}$$

From (3.21) it follows that $t \neq 0, 1$ and that

$$x = \cosh^*\alpha - \frac{\cosh^*\alpha\ \sinh\alpha}{\sinh\alpha - \sinh\beta} \quad \Rightarrow \quad x \neq 0,\ \cosh^*\alpha; \tag{3.23}$$

$$y = \frac{\cosh^*\beta\ \sinh\alpha}{\sinh\alpha - \sinh\beta} \quad \Rightarrow \quad y \neq 0,\ \cosh^*\beta. \tag{3.24}$$

Then

$$\frac{x}{\cosh^*\alpha} + \frac{y}{\cosh^*\beta} = 1. \tag{3.25}$$

From (3.24), (3.25) we obtain

$$\begin{aligned} \cosh^*\alpha &= \frac{x\cosh^*\beta}{\cosh^*\beta - y}, \\ \sinh\alpha &= \frac{y\sinh\beta}{y - \cosh^*\beta}, \end{aligned} \tag{3.26}$$

because, by (3.24), we know that $y \neq \cosh^*\beta$.

Next, since $(\cosh^*\alpha)^2 - \sinh^2\alpha = 1$, from (3.26) we have

$$x^2(\cosh^*\beta)^2 - y^2\sinh^2\beta = (y - \cosh^*\beta)^2. \tag{3.27}$$

Hence, simplifying the expression above, we find

$$\Big[(x^2 - y^2 - 1)\cosh^*\beta + 2y\Big]\cosh^*\beta = 0. \tag{3.28}$$

Since $\cosh^*\beta \neq 0$ and (by (3.24)) $y \neq 0$, we deduce that:

$$x^2 - y^2 - 1 \neq 0, \tag{3.29}$$

and then

$$\cosh^*\beta = \frac{-2y}{x^2 - y^2 - 1}. \tag{3.30}$$

Noting that $x \neq 0$, $\cosh^*\alpha$ (see (3.23)) by similar arguments we can derive that

$$y^2 - x^2 - 1 \neq 0 \tag{3.31}$$

and

$$\cosh^*\alpha = \frac{-2x}{y^2 - x^2 - 1}. \tag{3.32}$$

Finally, since (3.21) is equivalent to $|\cosh^*\alpha| > 1$, $|\cosh^*\beta| > 1$, from the expressions (3.30), (3.32) we deduce the conditions:

$$(i)\ \left|\frac{2y}{x^2 - y^2 - 1}\right| > 1 \quad \text{and} \quad (ii)\ \left|\frac{2x}{y^2 - x^2 - 1}\right| > 1. \tag{3.33}$$

Summing up, we have:

Claim 3.5 *If* (3.15), (3.21) *are verified and if* $P_3 = {}^t(x, y, 0)$ *is given by formula* (3.16), *then the necessary conditions* (3.29), (3.31) *and* (3.33) *are satisfied.*

Definition 3.6 *We will denote with* Σ *the subset of* $\mathbb{R}^2$ *where* (3.29), (3.31) *hold, i.e.,*

$$\Sigma \stackrel{\text{def}}{=} \big\{(x, y)\big|\ x^2 - y^2 \neq \pm 1\big\}. \tag{3.34}$$

The solution region of (3.33) is given by the following:

Lemma 3.7 *A pair* $(x, y) \in \Sigma$ *satisfies the conditions* (3.33) *(i) and (ii) iff*

$$|x| + |y| > 1 \quad \textit{and} \quad \big||x| - |y|\big| < 1 \tag{3.35}$$

or, equivalently,

$$(x + y + 1)(x + y - 1)(x - y + 1)(x - y - 1) < 0. \tag{3.36}$$

Proof. The inequalities of (3.33) is invariant under symmetry with respect to the coordinate axes, i.e., on replacing (x, y) with $(\pm x, \pm y)$. So it is sufficient to solve (3.33) for $x, y \geq 0$. Besides, we can obtain the first of (3.33) from the second, and vice versa, by permutation of the variables x, y. Hence it is sufficient to solve the second inequality of (3.33).

To begin with, for $(x, y) \in \Sigma$ with $x, y \geq 0$, inequality (3.33) (ii) is equivalent to

$$-2x < y^2 - x^2 - 1 < 2x, \tag{3.37}$$

that is

$$(x-1)^2 < y^2 < (x+1)^2. \tag{3.38}$$

which, in turn, is equivalent to

$$|x-1| < y < x+1, \tag{3.39}$$

because $x + 1 \geq 0$ and $y \geq 0$. Next, it easy to see that

$$\big\{(x, y) \,\big|\, |x-1| < y < x+1\big\} = \big\{(x, y) \,\big|\, x + y > 1,\ |x-y| < 1\big\} \subset \big\{x, y \geq 0\big\}. \tag{3.40}$$

Thus, for $x, y \geq 0$, the solution region of (3.33) (ii) is given by

$$\Omega \,=\, \Sigma \cap \big\{(x, y) \,\big|\, x + y > 1,\ |x-y| < 1\big\}. \tag{3.41}$$

The set Ω in (3.41) is symmetric with respect to x, y. By the previous considerations, Ω gives also the solution region of (3.33) (i) for $x, y \geq 0$ and, taking into account the symmetry with respect to the axes, from this we immediately obtain (3.35). Finally, it is easy to verify the equivalence of (3.35) and (3.36), because they define the same subset of $\mathbb{R} \times \mathbb{R}$. □

So far, we have proved that:

Claim 3.8 *If the conditions* (3.15), (3.21) *are verified and if* $P = {}^t(x, y, 0)$ *is given by* (3.16), *then* $(x, y) \in \Sigma$ *and*

$$g(x, y) \overset{\text{def}}{=} (x + y + 1)(x + y - 1)(x - y + 1)(x - y - 1) < 0.\,[31] \tag{3.42}$$

The converse is also true:

Claim 3.9 *If a point* $P = {}^t(x, y, 0)$ *is such that* $(x, y) \in \Sigma$ *and* (3.42) *holds, then* P *is given by formula* (3.16) *for suitable* α, β *satisfying* (3.15), (3.21).

Proof. Let us suppose that $(x, y) \in \Sigma$ satisfies (3.42). Then, by Lem. 3.7, there are (unique except for the sign) α, β such that

$$\cosh^*\alpha = \frac{-2x}{y^2 - x^2 - 1} \quad \text{and} \quad \cosh^*\beta = \frac{-2y}{x^2 - y^2 - 1}\,. \tag{3.43}$$

Since $|\cosh^* t| > 1 \Rightarrow \sinh t \neq 0$, condition (3.21) is certainly verified. With $\cosh^*\alpha$, $\cosh^*\beta$ such that (3.43) holds, the first two equations of (3.16) are satisfied by

$$t = \frac{y^2 - x^2 + 1}{2}\,. \tag{3.44}$$

[31] Note that condition (3.42) implies $x, y \neq 0$.

Then, with t as in (3.44), the third equation of (3.16) is verified iff

$$\frac{\sinh\beta}{\sinh\alpha} = -\frac{y^2-x^2-1}{x^2-y^2-1}. \tag{3.45}$$

Now, introducing the expressions (3.43) inside the identity $\sinh^2 t = (\cosh^* t)^2 - 1$, we obtain

$$\sinh^2\alpha = -\frac{g(x,y)}{(y^2-x^2-1)^2} \quad \text{and} \quad \sinh^2\beta = -\frac{g(x,y)}{(x^2-y^2-1)^2}, \tag{3.46}$$

where $g(x,y)$ is the quantity defined in (3.42). Since we are assuming $g(x,y) < 0$, we may conclude that (3.45) holds iff

$$(\sinh\alpha, \sinh\beta) = \pm\left(\frac{\sqrt{-g(x,y)}}{y^2-x^2-1}, \frac{-\sqrt{-g(x,y)}}{x^2-y^2-1}\right). \tag{3.47}$$

Finally, it remains to note that for $(x,y)\in\Sigma$ the relation (3.45) gives also the inequality $\sinh\alpha \neq \sinh\beta$, i.e., condition (3.15). In conclusion, we have proved that there are α, β such that both conditions (3.15), (3.21) hold and $P = {}^t(x,y,0)$ satisfies formula (3.16). □

Recalling (3.13), (3.43) and (3.47), we may conclude the following:

Claim 3.10 *Let us suppose $x, y \neq 0$. Then system* (3.16) *with condition* (3.15) *is solvable* $\Leftrightarrow$ $(x,y)\in\Sigma$ *and* (3.42) *holds. Moreover, if $(x,y)\in\Sigma$ and* (3.42) *holds, then*

$$\begin{aligned} Q_3 &= \frac{1}{y^2-x^2-1}\begin{pmatrix} -2x \\ 0 \\ \delta\sqrt{-g(x,y)} \end{pmatrix} \\ & \qquad\qquad\qquad\qquad\qquad \textit{with} \quad \delta = \pm 1. \\ Q_3' &= \frac{1}{y^2-x^2+1}\begin{pmatrix} 0 \\ 2y \\ \delta\sqrt{-g(x,y)} \end{pmatrix} \end{aligned} \tag{3.48}$$

Proof. As we have already observed at the beginning of section 3.4,

$$x, y \neq 0 \text{ and } (3.15), (3.16) \implies (3.21).$$

Therefore, it is sufficient to apply Claim 3.8 and Claim 3.9. □

The previous statement gives the necessary and sufficient conditions for the existence of Q_3, Q_3' such that (a), (b), (c) of Cond. 3.3 hold, i.e., such that there is a projection $\Pi_{\mathbf{v}}$ satisfying (1.15) and (1.16) of Def. 1.4, when (3.3) holds and $P_3 = {}^t(x,y,0)$ with $x, y \neq 0$.
So, in order to have a hyperbolic Pohlke's projection, it only remains to verify if (d) of Cond. 3.3 holds when Q_3, Q_3' are given by (3.48). To this end, noting (3.13), we write:

$$\mathbf{v} = \overrightarrow{Q_3Q_3'} = -(\cosh^*\alpha)\mathbf{i} + (\cosh^*\beta)\mathbf{j} + (\sinh\beta - \sinh\alpha)\mathbf{k}, \tag{3.49}$$

with $\cosh^*\alpha$, $\cosh^*\beta$ as in (3.43) and $\sinh\alpha$, $\sinh\beta$ as in (3.47). Then we have:

Claim 3.11 *Let* $P_3 = {}^t(x, y, 0)$ *with* $(x, y) \in \Sigma$ *such that* (3.42) *holds. Then the projection direction* $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ *given by* (3.49) *satisfies*

$$l^2 + m^2 - n^2 = 4\,\frac{x^2 + y^2 - 1}{(x^2 - y^2)^2 - 1}. \tag{3.50}$$

Proof. Assuming $(x, y) \in \Sigma$ and (3.42), the expressions (3.43) and (3.47) are well defined real numbers. Then writing $\mathbf{v}$ as in (3.49) and using (3.47), we find that

$$\begin{aligned} l^2 + m^2 - n^2 &= (\cosh^*\alpha)^2 + (\cosh^*\beta)^2 - (\sinh\beta - \sinh\alpha)^2 \\ &= 2\big(1 + \sinh\alpha \sinh\beta\big) \\ &= 2\left[1 + \frac{g(x, y)}{(y^2 - x^2 - 1)(x^2 - y^2 - 1)}\right] \\ &= 2\left[1 - \frac{g(x, y)}{(x^2 - y^2)^2 - 1}\right] \\ &= 4\,\frac{x^2 + y^2 - 1}{(x^2 - y^2)^2 - 1}. \qquad \square \end{aligned} \tag{3.51}$$

Finally, taking into account Rem. 3.1, Claim 3.2, Cond.3.3 and summing up, we have:

Lemma 3.12 *If* (3.3) *is verified and if* $OP_3 \nparallel OP_1, OP_2$*, then there is a hyperbolic Pohlke's projection* $\Pi_{\mathbf{v}}$ *for* OP_1, OP_2, OP_3 *if and only if*

$$\overrightarrow{OP_3} = x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}, \tag{3.52}$$

with (x, y) *such that* (3.42) *holds and*

$$f(x, y) \stackrel{\text{def}}{=} (x^2 + y^2 - 1)(x^2 - y^2 - 1)(x^2 - y^2 + 1) \neq 0. \tag{3.53}$$

If the conditions (3.42) *and* (3.53) *are verified, then the hyperbolic Pohlke's projection* $\Pi_{\mathbf{v}}$ *is unique up to symmetry with respect to the plane* ω*. The conic* $\mathscr{C}_{\mathbf{v}}$ *is unique and* $\mathscr{C}_{\mathbf{v}}$ *is an ellipse if* $f(x, y) < 0$*, while* $\mathscr{C}_{\mathbf{v}}$ *is a hyperbola if* $f(x, y) > 0$*.*

Proof. Let us first note that

$$f(x, y) \neq 0 \quad\Leftrightarrow\quad (x, y) \in \Sigma \quad\text{and}\quad x^2 + y^2 \neq 1. \tag{3.54}$$

Suppose now that (3.42), (3.53) are true. Then the existence of a projection $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$ satisfying (1.15), (1.16) of Def. 1.4 follows from Claim 3.10. Thanks to Claim 3.11 and (3.54), the condition $f(x, y) \neq 0$ also implies that (d) of Cond. 3.3 is true, i.e., $\Pi_{\mathbf{v}}$ is non-degenerate. Hence $\Pi_{\mathbf{v}}$ is a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3.

Conversely, let $\Pi_{\mathbf{v}}$ be a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3. Taking into account Cond. 3.3 and the arguments at the beginning of Section 3.2, we deduce from Claim 3.10 that $(x, y) \in \Sigma$ and (3.42) holds. Furthermore, the points Q_3, Q_3' are necessarily given by (3.48). Since $\Pi_{\mathbf{v}}$ is non-degenerate, (3.50) of Claim. 3.11 gives $x^2 + y^2 \neq 1$. By (3.54) we can finally see that also (3.53) holds.

As for the uniqueness of $\Pi_{\mathbf{v}}$, we recall that by Claim 3.2 we necessarily have $\rho = 1$, that is $\mathscr{H} = \mathscr{H}(1)$. Furthermore, the vector $\mathbf{v} = \overrightarrow{Q_3Q_3'}$, given by Claim 3.10, is uniquely determined up to choosing the plus and minus sign in formula (3.48). This means that we can obtain only two projections, $\Pi_{\mathbf{v}}$ and $\bar{\Pi}_{\mathbf{v}}$, which are symmetric with respect to the plane ω (according to the second part of Rem. 3.1). Hence, taking into account that $\mathscr{H} = \mathscr{H}(1)$, from (3.2) we may conclude that $\mathscr{C}_{\mathbf{v}} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}})$ is unique. Finally, assuming that the hyperbolic Pohlke's projection exists, by Cor. 2.11 and (3.50) above, the conic $\mathscr{C}_{\mathbf{v}}$ is an ellipse or a hyperbola depending on whether it is $f(x,y) > 0$ or $f(x,y) < 0$. □

4 Proof of Theorem 1.8

(1) ⇒ (2). It is sufficient to apply part (1) of Claim 2.30 first and then Cor. 2.13.
Indeed, since we are assuming $OP_i \nparallel OP_j$ $(1 \leq i < j \leq 3)$ by the conditions (1.15), (1.16) of Def. 1.4 and part (1) of Claim 2.30, we have:

$$\Pi_{\mathbf{v}}(Q_1) = P_1,\ \Pi_{\mathbf{v}}(Q_2) = P_2 \quad \text{and} \quad OQ_1 \parallel T_{\mathscr{H}}(Q_2) \ \Rightarrow\ \Pi_{\mathbf{v}}(\mathscr{C}(Q_1,Q_2)) = \mathcal{E}_{P_1,P_2}, \tag{4.1}$$

$$\Pi_{\mathbf{v}}(Q_2) = P_2,\ \Pi_{\mathbf{v}}(Q_3) = P_3 \quad \text{and} \quad OQ_2 \parallel T_{\mathscr{H}}(Q_3) \ \Rightarrow\ \Pi_{\mathbf{v}}(\mathscr{C}(Q_2,Q_3)) = \mathcal{E}_{P_2,P_3} \tag{4.2}$$

and, noting that $\Pi_{\mathbf{v}}(Q_1') = P_1$,

$$\Pi_{\mathbf{v}}(Q_3) = P_3,\ \Pi_{\mathbf{v}}(Q_1') = P_1 \quad \text{and} \quad OQ_3 \parallel T_{\mathscr{H}}(Q_1') \ \Rightarrow\ \Pi_{\mathbf{v}}(\mathscr{C}(Q_3,Q_1')) = \mathcal{E}_{P_3,P_1}. \tag{4.3}$$

Furthermore, $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ are tangent to

$$\mathscr{C}_{\mathbf{v}} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}). \tag{4.4}$$

Then, since $\mathcal{E}_{P_1,P_2}, \mathcal{E}_{P_2,P_3}, \mathcal{E}_{P_3,P_1} \subset \Pi_{\mathbf{v}}(\mathscr{H})$, by Cor. 2.13 we finally deduce that:

- $\mathscr{C}_{\mathbf{v}}$ is inscribed in $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ if $\mathscr{C}_{\mathbf{v}}$ is an ellipse;
- $\mathscr{C}_{\mathbf{v}}$ circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$ if $\mathscr{C}_{\mathbf{v}}$ is a hyperbola.

In conclusion $\mathcal{C} = \mathscr{C}_{\mathbf{v}}$ is a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3.

(2) ⇒ (1). This implication can be obtained by first applying Claim 2.15, Rem. 2.17 and then Claim 2.19 and Claim A.3 of the Appendix.
Indeed, let $\mathcal{C}$ be a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3. We fix $\mathscr{H} = \mathscr{H}(\rho)$ with

$$\rho = \text{major/transverse semi-axis of } \mathcal{C} \quad (\rho = \text{radius, if } \mathcal{C} \text{ is a circle}). \tag{4.5}$$

Then from the three cases of Rem. 2.17 we obtain, up to symmetry with respect to the plane ω, the projection direction, i.e., the vector $\mathbf{v}$. Moreover, by Claim 2.18, $\mathbf{v}$ is non-degenerate.
This means that we can realized $\mathcal{C}$ as a projection of a section the hyperboloid $\mathscr{H} = \mathscr{H}(\rho)$. More precisely, we have:

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}) \overset{\text{def}}{=} \mathscr{C}_{\mathbf{v}}. \tag{4.6}$$

After that, we consider $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$, which are tangent to $\mathscr{C}_{\mathbf{v}}$, by Def. 1.7. Starting with $\mathcal{E}_{P_1,P_2}$, by 1) of Claim 2.19 there is a plane π, through the origin O, such that $\mathscr{H} \cap \pi$ is an ellipse and $\Pi_{\mathbf{v}}(\mathscr{H} \cap \pi) = \mathcal{E}_{P_1,P_2}$.[32] Then, by 2) of Claim 2.19, there are $Q_1, Q_2 \in \mathscr{H} \cap \pi$

[32] From (2.36) we know that $\Pi_{\mathbf{v}}\big|_{\pi} : \pi \to \omega$ is an affine transformation. Hence $\mathscr{H} \cap \pi$ must be an ellipse.

such that $\Pi(Q_1) = P_1$, $\Pi(Q_2) = P_2$ and OQ_1, OQ_2 are conjugate semi-diameters of the ellipse $\mathscr{H} \cap \pi$. With the notation of Def. 2.27, this later fact implies $OQ_1 \parallel T_{\mathscr{C}(Q_1,Q_2)}(Q_2)$. Then

$$OQ_1 \parallel T_{\mathscr{C}(Q_1,Q_2)}(Q_2) \quad \text{and} \quad T_{\mathscr{C}(Q_1,Q_2)}(Q_2) \subset T_{\mathscr{H}}(Q_2) \;\Rightarrow\; OQ_1 \parallel T_{\mathscr{H}}(Q_2). \tag{4.7}$$

So the first condition of (1.16) is satisfied. To proceed further, we consider $\mathcal{E}_{P_2,P_3}$. Again from 1) and 2) of Claim 2.19 we can find a plane $\widetilde{\pi}$, through O and Q_2, such that $\mathscr{H} \cap \widetilde{\pi}$ is an ellipse and $\Pi_{\mathbf{v}}(\mathscr{H} \cap \widetilde{\pi}) = \mathcal{E}_{P_2,P_3}$. Besides, we can also find a point $Q_3 \in \mathscr{H} \cap \widetilde{\pi}$ such that $\Pi(Q_3) = P_3$ and OQ_2, OQ_3 are conjugate semi-diameters of $\mathscr{H} \cap \widetilde{\pi}$. As above, we deduce that

$$OQ_2 \parallel T_{\mathscr{H}}(Q_3). \tag{4.8}$$

So, the second condition of (1.16) holds. Finally, we consider the ellipse $\mathcal{E}_{P_3,P_1}$. Noting that

$$\Pi^{-1}(P_1) \cap \mathscr{H} = \{Q_1, Q_1'\}, \tag{4.9}$$

and reasoning as above, it is clear that at least one of the following must be true:

$$OQ_3 \parallel T_{\mathscr{H}}(Q_1) \quad \text{or} \quad OQ_3 \parallel T_{\mathscr{H}}(Q_1'). \tag{4.10}$$

But, by Claim A.3, we cannot have the sequence

$$OQ_1 \parallel T_{\mathscr{H}}(Q_2),\; OQ_2 \parallel T_{\mathscr{H}}(Q_3) \;\text{ and }\; OQ_3 \parallel T_{\mathscr{H}}(Q_1), \tag{4.11}$$

with $Q_1, Q_2, Q_3 \in \mathscr{H}$. Hence the second (and only the second) of (4.10) is true. In conclusion, we have found $Q_1, Q_2, Q_3 \in \mathscr{H}$ such that (1.15) and (1.16) hold.

4.1 The equivalence of (1), (2) with (3)

To prove that (1),(2) $\Leftrightarrow$ (3) when OP_1, OP_2, OP_3 are non-parallel, we resort to an appropriate circular case. More precisely, let $N_1, N_2 \in \omega$ such that

$$ON_1 \perp ON_2 \quad \text{and} \quad |ON_1| = |ON_2| = 1. \tag{4.12}$$

Since $OP_1 \nparallel OP_2$, we may consider the affine transformation $\Phi : \omega \to \omega$ defined by

$$\Phi(O + x\,\overrightarrow{OP_1} + y\,\overrightarrow{OP_2}\,) \overset{\text{def}}{=} O + x\,\overrightarrow{ON_1} + y\,\overrightarrow{ON_2} \quad \text{for} \quad x, y \in \mathbb{R}. \tag{4.13}$$

It is clear that $\Phi(P_1) = N_1$, $\Phi(P_2) = N_2$. Besides, if $\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2}$, then

$$N_3 \overset{\text{def}}{=} \Phi(P_3) = O + h\,\overrightarrow{ON_1} + k\,\overrightarrow{ON_2}. \tag{4.14}$$

Hence

$$\overrightarrow{ON_3} = h\,\overrightarrow{ON_1} + k\,\overrightarrow{ON_2} \quad \text{and} \quad ON_3 \nparallel ON_1, ON_2, \tag{4.15}$$

because $OP_3 \nparallel OP_1, OP_2$ (i.e., $h, k \neq 0$).

As it is known, an affine transformation maps conjugate semi-diameters of a central conic into conjugate semi-diameters of the transformed conic.[14] This means that $\Phi(\mathcal{E}_{P_1,P_2}) = \mathcal{E}_{N_1,N_2}$, $\Phi(\mathcal{E}_{P_2,P_3}) = \mathcal{E}_{N_2,N_3}$ and $\Phi(\mathcal{E}_{P_3,P_1}) = \mathcal{E}_{N_3,N_1}$. Besides, if $\mathcal{C}$ is a hyperbola (ellipse), with center O, which circumscribes (is inscribed in) $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$, then $\Phi(\mathcal{C})$ is a hyperbola (ellipse) centered at O which circumscribes (is inscribed in) $\mathcal{E}_{N_1,N_2}$, $\mathcal{E}_{N_2,N_3}$, $\mathcal{E}_{N_3,N_1}$.

The converse is also true, because $\Phi^{-1} : \omega \to \omega$ is still an affine transformation. Hence, according to Def. 1.7, we can state the following:

Claim 4.1 *If $\mathcal{C}$ is a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3, then $\Phi(\mathcal{C})$ is a hyperbolic Pohlke's conic for ON_1, ON_2, ON_3, and vice versa.*

(1),(2) ⇒ (3). Now let us suppose that (2) holds, namely that there is a hyperbolic Pohlke's conic $\mathcal{C}$ for OP_1, OP_2, OP_3. Then

$$\mathcal{C}_o = \Phi(\mathcal{C}) \tag{4.16}$$

is a hyperbolic Pohlke's conic for ON_1, ON_2, ON_3. Hence, having already proved that (1) ⇔ (2), there is a hyperbolic Pohlke's projection for ON_1, ON_2, ON_3. By (4.12) and (4.15) we can therefore apply Lem. 3.12 to ON_1, ON_2, ON_3. Thus we conclude that h, k must satisfy the conditions (1.17) and (1.18).

(3) ⇒ (1),(2). Conversely, let us suppose that (3) hold, i.e., h, k satisfy the conditions (1.17) and (1.18). Then, by Lem. 3.12, there is a hyperbolic Pohlke's projection for ON_1, ON_2, ON_2. By the equivalence (1) ⇔ (2), we deduce the existence of a hyperbolic Pohlke's conic, say $\mathcal{C}_o$, for ON_1, ON_2, ON_3. Then,

$$\mathcal{C} = \Phi^{-1}(\mathcal{C}_o) \tag{4.17}$$

is a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3. Thus we have proved that (2) holds.

4.2 Uniqueness of $\Pi_{\mathbf{v}}, \mathcal{C}$ and conic type of $\mathcal{C}$

The uniqueness properties of hyperbolic Pohlke's conic $\mathcal{C}$ and of hyperbolic Pohlke's projection $\Pi_{\mathbf{v}}$ follow immediately from the circular case studied in Section 3. In fact, if we assume condition (3.3), by Claim 3.2 we necessarily have $\rho = 1$, that is $\mathscr{H} = \mathscr{H}(1)$. Besides, by Lem. 3.12, the projection direction, given by the vector $\overrightarrow{Q_3Q_3'}$ in (3.49), is unique up to symmetry with respect to the plane ω. That is, we have:

$$\mathbf{v} \parallel \mathbf{v}_+ \ \text{ or } \ \mathbf{v} \parallel \mathbf{v}_- \quad \text{with} \quad \mathbf{v}_\pm = l\mathbf{i} + m\mathbf{j} \pm n\mathbf{k}, \tag{4.18}$$

for suitable l, m, n such that $n \neq 0$ and $l^2 + m^2 - n^2 \neq 0$. Therefore we have the uniqueness of the hyperbolic Pohlke's conic in the circular case, because

$$\mathcal{C} = \Pi_{\mathbf{v}_+}(\mathscr{H} \cap \pi_{\mathbf{v}_+}) = \Pi_{\mathbf{v}_-}(\mathscr{H} \cap \pi_{\mathbf{v}_-}). \tag{4.19}$$

Having proved the uniqueness in the circular case, applying the affine transformation $\Phi : \omega \to \omega$ introduced un (4.13), we deduce the uniqueness of the hyperbolic Pohlke's conic in general. As for the hyperbolic Pohlke's projection $\Pi_{\mathbf{v}} : \mathbb{R}^3 \to \omega$, it is enough to recall that the hyperbolic Pohlke's conic $\mathcal{C}$ uniquely determines the hyperboloid $\mathscr{H} = \mathscr{H}(\rho)$ and, up to symmetry with respect to the plane ω, the projection direction $\mathbf{v}$. See Rem. 2.17.

Finally, since $\mathcal{C}$ and $\Phi(\mathcal{C})$ are conic of the same type, by Lem. 3.12 it is clear that the hyperbolic Pohlke's conic $\mathcal{C}$ is an ellipse if $f(h,k) < 0$, while it is a hyperbola if $f(h,k) > 0$.

5 Proof of Theorem 1.11

We will first show that the equivalence (1) ⇔ (2) of Thm. 1.8 remains valid under the hypotheses of Thm. 1.11, if we allow degenerate ellipses, in the sense of Def. 1.9, and if we replace Def. 1.7 with Def. 1.10 as hyperbolic Pohlke's conic definition.

According to the hypotheses, we will assume that OP_1, OP_2, OP_3 are not contained in a line, but two of them are parallel to each other. More precisely, in the following we will suppose that

$$OP_1 \nparallel OP_2 \quad \text{and} \quad OP_2 \parallel OP_3. \tag{5.1}$$

(1) ⇒ (2). We apply part (1) of Claim 2.30 (as in the proof of Thm. 1.8) if $OP_i \nparallel OP_j$, and part (2) of Claim 2.30 if $OP_i \parallel OP_j$. To begin with, since we suppose $OP_1 \nparallel OP_2$, by the conditions (1.15), (1.16) of Def. 1.4 and part (1) of Claim 2.30 we deduce that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_1, Q_2)) = \mathcal{E}_{P_1,P_2} \quad \text{and } \mathscr{C}_{\mathbf{v}} \text{ is tangent to } \mathcal{E}_{P_1,P_2}. \tag{5.2}$$

To proceed, we consider then the pair OP_2, OP_3. In this case $OP_2 \parallel OP_3$, thus $\mathcal{E}_{P_2,P_3}$ is a degenerate ellipse in the sense of Def. 1.9. Hence, by part (2) of Claim 2.30, we deduce that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_2, Q_3)) = \mathcal{E}_{P_2,P_3} \quad \text{and that } \mathscr{C}_{\mathbf{v}} \text{ is a hyperbola circumscribing } \mathcal{E}_{P_2,P_3}.$$

Knowing that $\mathscr{C}_{\mathbf{v}}$ is a hyperbola, from (5.2) and Cor. 2.13 it also follows that $\mathscr{C}_{\mathbf{v}}$ circumscribes the ellipse $\mathcal{E}_{P_1,P_2}$. Finally, we consider the pair OP_3, OP_1. Applying as above (1) of Claim 2.30 (if $OP_3 \nparallel OP_1$) or (2) of Claim 2.30 (if $P_3 = O$), we find that

$$\Pi_{\mathbf{v}}(\mathscr{C}(Q_3, Q_1')) = \mathcal{E}_{P_3,P_1} \quad \text{and } \mathscr{C}_{\mathbf{v}} \text{ circumscribes } \mathcal{E}_{P_3,P_1}. \tag{5.3}$$

In conclusion, we have proved that $\mathscr{C}_{\mathbf{v}}$ is a hyperbola circumscribing $\mathcal{E}_{P_1,P_2}, \mathcal{E}_{P_2,P_3}, \mathcal{E}_{P_3,P_1}$. Hence $\mathscr{C}_{\mathbf{v}}$ is a hyperbolic Pohlke's conic, in the sense of Def. 1.10, for OP_1, OP_2, OP_3.

(2) ⇒ (1). Let $\mathcal{C}$ be a hyperbolic Pohlke's conic in the sense of Def. 1.10. By applying Claim 2.15 and Rem. 2.17 (as in the proof of Thm. 1.8) we can determine the hyperboloid $\mathscr{H} = \mathscr{H}(\rho)$ and the projection direction, represented by $\mathbf{v}$, up to symmetry with respect to the plane ω. It automatically follows that $\mathbf{v}$ is non-degenerate (by Claim 2.18) and that

$$\mathcal{C} = \Pi_{\mathbf{v}}(\mathscr{H} \cap \pi_{\mathbf{v}}) \stackrel{\text{def}}{=} \mathscr{C}_{\mathbf{v}}.$$

After this we consider the (eventually degenerate) ellipses $\mathcal{E}_{P_1,P_2}, \mathcal{E}_{P_2,P_3}, \mathcal{E}_{P_3,P_1}$. Using 1) and 2) of Claim 2.19 (if $OP_i \nparallel OP_j$) or Claim 2.23 (if $OP_i \parallel OP_j$) and then Claim A.3, we can show that there are $Q_1, Q_2, Q_3 \in \mathscr{H}$ such that the conditions (1.15), (1.16) of Def. 1.4 are verified. In this way we prove that $\Pi_{\mathbf{v}}$ is a hyperbolic Pohlke's projection for OP_1, OP_2, OP_3.

Conclusion of the proof. We can now prove that under the assumptions (5.1) the are infinite, distinct hyperbolic Pohlke's projections (conics) if $|OP_2| = |OP_3|$, none if $|OP_2| \neq |OP_3|$.
To this end, we resort to the circular case as in the proof of Thm. 1.8. Namely, since we assume $OP_1 \nparallel OP_2$, we may consider the affine transformation $\Phi : \omega \to \omega$ defined in (4.13). In this case we have $\Phi(P_i) = N_i$, for $1 \leq i \leq 3$, with

$$ON_1 \perp ON_2, \quad |ON_1| = |ON_2| = 1 \quad \text{and} \quad ON_2 \parallel ON_3. \tag{5.4}$$

We note also that Claim 4.1 continues to hold even though we apply Def. 1.10 instead of Def. 1.7. So we still have that $\mathcal{C}$ is a hyperbolic Pohlke's conic for OP_1, OP_2, OP_3 if and only if $\Phi(\mathcal{C})$ is a hyperbolic Pohlke's conic for ON_1, ON_2, ON_3. [33]

[33] It is worth noting that if $\mathcal{E}_{P,Q}$ is a degenerate ellipse, then $\Phi(\mathcal{E}_{P,Q}) = \mathcal{E}_{\Phi(P),\Phi(Q)}$.

Now, having $ON_2 \parallel ON_3$, by Lem. 3.4 there are infinite, distinct hyperbolic Pohlke's projections for ON_1, ON_2, ON_3 if $|ON_3| = 1$, none if $|ON_3| \neq 1$. By the equivalence (1) $\Leftrightarrow$ (2) proved above, it follows that there are infinite, distinct hyperbolic Pohlke's conics for ON_1, ON_2, ON_3 if $|ON_3| = 1$, none if $|ON_3| \neq 1$. Since

$$|ON_3| = 1 \quad \Leftrightarrow \quad |OP_3| = |OP_2|, \tag{5.5}$$

we deduce that, under assumption (5.1), there are infinite, distinct hyperbolic Pohlke's conics for OP_1, OP_2, OP_3 if $|OP_3| = |OP_2|$, none if $|OP_3| \neq |OP_2|$. Finally, again by the equivalence (1) $\Leftrightarrow$ (2), the same holds for the hyperbolic Pohlke's projections.

A Appendix

A.1 Affine transformations of conics

By affine transformation between two planes $\pi, \tilde{\pi} \subset \mathbb{E}^3$ we mean here any map $T : \pi \to \tilde{\pi}$ such that if x, y and $\tilde{x}, \tilde{y}$ are coordinates in π and $\tilde{\pi}$ respectively, and if $P \in \pi$ has coordinates $p = (x, y)$, then the coordinates $\tilde{p} = (\tilde{x}, \tilde{y})$ of $\tilde{P} = T(P)$ are given by

$$\tilde{p} = pA + q, \tag{A.1}$$

where A is a 2×2, invertible matrix and $q = (q_1, q_2)$ is constant.

It is an elementary fact that affine transformations send a conic to a conic of the same type, i.e. the *structure* of the conic is preserved:

Theorem A.1 *Let $T : \pi \to \tilde{\pi}$ be an affine transformation between the planes $\pi, \tilde{\pi}$. Then T maps an ellipse to an ellipse, a parabola to a parabola, a hyperbola to a hyperbola and a degenerate conic to a degenerate conic of the same type.*

Proof. If $\mathcal{C} \subset \pi$ is a degenerate (i.e., reducible) conic it is clear that $T(\mathcal{C}) \subset \tilde{\pi}$ is a degenerate conic of the same type, because T is merely the composition of an invertible linear map with a translation in the plane $\tilde{\pi}$. If $\mathcal{C} \subset \pi$ is a non-degenerate we refer to [1, Thm. 12.12]. □

We use this fact many times in this paper, specially in the case where T is a parallel projection in the direction of a given vector $\mathbf{v}$ such that $\mathbf{v} \nparallel \pi, \tilde{\pi}$, i.e.,

$$T(P) = \tilde{\pi} \cap \{P + t\mathbf{v} \,|\, t \in \mathbb{R}\} \quad \text{for} \quad P \in \pi. \tag{A.2}$$

Remark A.2 *We also note, omitting the standard proof, that:*

1) *If the segments OP and OQ are conjugate semi-diameters of an ellipse $\mathcal{E} \subset \pi$, then $T(OP)$ and $T(OQ)$ are conjugate semi-diameters of the ellipse $T(\mathcal{E}) \subset \tilde{\pi}$.*

2) *With a slight abuse of language we can also talk about conjugate semi-diameters of a hyperbola $\mathcal{H}$: if the axes passing through OP and OQ are conjugate diameters, we say that the segments OP, OQ are conjugate semi-diameters, with OP transverse, if in the (generally) oblique coordinate system (x, y) given by these axes the equation of $\mathcal{H}$ is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{A.3}$$

with $a, b > 0$ such that the coordinates of P and Q are $(\pm a, 0)$ and $(0, \pm b)$, respectively. In this situation we could say that OP is the transverse semi-diameter and that OQ is the corresponding "imaginary" conjugate semi-diameter. As in the case of the ellipse, if the segments OP and OQ are conjugate semi-diameters of the hyperbola $\mathcal{H} \subset \pi$, it turns out that $T(OP)$ and $T(OQ)$ are conjugate semi-diameters of the hyperbola $T(\mathcal{H}) \subset \tilde{\pi}$.

A.2 On the definition of hyperbolic Pohlke's projection

In Def. 1.4 it may seem more natural to require the condition

$$OQ_1 \parallel T_{\mathscr{H}}(Q_2),\ OQ_2 \parallel T_{\mathscr{H}}(Q_3)\ \text{ and }\ OQ_3 \parallel T_{\mathscr{H}}(Q_1), \tag{A.4}$$

rather than (1.16), i.e.,

$$OQ_1 \parallel T_{\mathscr{H}}(Q_2),\ OQ_2 \parallel T_{\mathscr{H}}(Q_3)\ \text{ and }\ OQ_3 \parallel T_{\mathscr{H}}(Q_1'), \tag{A.5}$$

where $Q_1' \in \mathscr{H}(\rho)$ is the point $\pi_{\mathbf{v}} - symmetric$ to Q_1. But, if we replace condition (A.5) with (A.4), then Def. 1.4 does not work.

Claim A.3 *There does not exist $Q_1, Q_2, Q_3 \in \mathscr{H}(\rho)$ such that* (A.4) *holds.*

Proof. In fact, writing $Q_1 = (x_1, y_1, z_1)$, $Q_2 = (x_2, y_2, z_2)$, $Q_3 = (x_3, y_3, z_3)$, by (2.59) we can reformulate (A.4) in the equivalent form:

$$\begin{cases} x_1x_2 + y_1y_2 - z_1z_2 = 0 \\ x_2x_3 + y_2y_3 - z_2z_3 = 0 \\ x_1x_3 + y_1y_3 - z_1z_3 = 0 \end{cases} \tag{A.6}$$

Then, assuming $Q_1, Q_2 \in \mathscr{H}(\rho)$ are such that $OQ_1 \parallel T_{\mathscr{H}}(Q_2)$ (i.e., the first equation of (A.6) holds), we can show that there does not exist $Q_3 \in \mathscr{H}(\rho)$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$ and $OQ_3 \parallel T_{\mathscr{H}}(Q_1)$ (i.e., the last two equations of (A.6) hold).

By contradiction let us suppose that such a point Q_3 exists. Noting that $OQ_1 \nparallel OQ_2$ (see Rem. 2.28), from the last two equations of (A.6), we deduce that:

$$x_3 = \lambda \begin{vmatrix} y_1 & -z_1 \\ y_2 & -z_2 \end{vmatrix}, \quad y_3 = -\lambda \begin{vmatrix} x_1 & -z_1 \\ x_2 & -z_2 \end{vmatrix}, \quad z_3 = \lambda \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \tag{A.7}$$

for a suitable $\lambda \neq 0$. To proceed, it is not restrictive to assume that the coordinate axes are chosen such that $Q_1 = (x_1, 0, z_1)$, that is,

$$y_1 = 0. \tag{A.8}$$

Hence (A.7) and (A.8) give

$$x_3 = \lambda z_1 y_2, \quad y_3 = \lambda\big(x_1 z_2 - z_1 x_2\big), \quad z_3 = \lambda x_1 y_2 \tag{A.9}$$

and then

$$x_3^2 + y_3^2 - z_3^2 = \lambda^2 \Big[z_1^2 y_2^2 + \big(x_1 z_2 - z_1 x_2\big)^2 - x_1^2 y_2^2\Big]. \tag{A.10}$$

Now, we observe that

$$\big(z_1^2 - x_1^2\big)y_2^2 = -\rho^2 y_2^2 \quad \text{because} \quad x_1^2 - z_1^2 = \rho^2. \tag{A.11}$$

So, if $x_2 = z_2 = 0$, from (A.10) and (A.11) we immediately obtain

$$x_3^2 + y_3^2 - z_3^2 = -\lambda^2\rho^2 y_2^2 = -\lambda^2\rho^4 < 0.^{34} \tag{A.12}$$

Since we must have $x_3^2 + y_3^2 - z_3^2 = \rho^2$, the inequality (A.12) gives a contradiction. Conversely, let us suppose $(x_2, z_2) \neq (0,0)$. With $y_1 = 0$ the first equation of (A.6) reads

$$\begin{vmatrix} x_1 & z_1 \\ z_2 & x_2 \end{vmatrix} = 0. \tag{A.13}$$

Having assumed $(x_2, z_2) \neq (0,0)$, we can deduce that

$$x_1 = \mu z_2, \quad z_1 = \mu x_2 \quad \text{for a suitable} \quad \mu \neq 0. \tag{A.14}$$

This means that

$$\big(x_1 z_2 - z_1 x_2\big)^2 = \mu^2\big(z_2^2 - x_2^2\big)^2. \tag{A.15}$$

On the other hand, since $x_1^2 - z_1^2 = \rho^2$, from (A.14) we also have

$$\mu^2(z_2^2 - x_2^2) = \rho^2. \tag{A.16}$$

Taking into account (A.15), we therefore find

$$\big(x_1 z_2 - z_1 x_2\big)^2 = \rho^2\big(z_2^2 - x_2^2\big) = \rho^2\big(y_2^2 - \rho^2\big), \tag{A.17}$$

because $x_2^2 + y_2^2 - z_2^2 = \rho^2$. Finally, from (A.10), (A.11) and (A.17), we obtain

$$x_3^2 + y_3^2 - z_3^2 = \lambda^2\Big[-\rho^2 y_2^2 + \rho^2\big(y_2^2 - \rho^2\big)\Big] = -\lambda^2\rho^4 < 0, \tag{A.18}$$

which gives the same contradiction of (A.12). □

In the case of the Def. 1.4, Claim A.3 has the following consequence:

Corollary A.4 *If the points $Q_1, Q_2, Q_3 \in \mathscr{H}$ satisfy the conditions of Def. 1.4 then $Q_i \neq Q_i'$ (that is $Q_i \notin \pi_{\mathbf{v}}$) for $\leq i \leq 3$.*

A.3 Examples of Hyperbolic Pohlke's projection

For completeness we report the calculations necessary to obtain the examples Ex. 1.12 and Ex. 1.13. After which, in Rem. A.5 below, we will make some general considerations.

Ex. 1.12 (detailed calculations). To begin with, we set $\rho = 1$ and $Q_1(2, 0, -\sqrt{3})$.

Choice of Q_2. We look for $Q_2(x_2, y_2, z_2) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$. With $Q_1(2, 0, -\sqrt{3})$, this is equivalent to require:

$$x_2^2 + y_2^2 - z_2^2 = 1 \quad \text{and} \quad 2x_2 + \sqrt{3}z_2 = 0, \tag{A.19}$$

[34] Note that $x_2 = z_2 = 0 \;\Rightarrow\; y_2^2 = \rho^2$.

that is, $4y_2^2 = 4 + z_2^2$ and $2x_2 = -\sqrt{3}z_2$. We can take $z_2 = 2$, $y_2 = \sqrt{2}$ and $x_2 = -\sqrt{3}$. So we have the point $Q_2(-\sqrt{3}, \sqrt{2}, 2)$.

Choice of Q_3^*. We choose $Q_3^*(x_3^*, y_3^*, z_3^*)$ by imposing the same conditions as Q_2:

$$4{y_3^*}^2 = 4 + {z_3^*}^2, \quad 2x_3^* = -\sqrt{3}z_3^*. \tag{A.20}$$

This time we can choose $z_3^* = -\frac{3}{2}$, $y_3^* = \frac{5}{4}$ and $x_3^* = -\frac{3\sqrt{3}}{4}$. Thus, we have $Q_3^*(\frac{3\sqrt{3}}{4}, \frac{5}{4}, -\frac{3}{2})$.

Choice of Q_3. After Q_2, Q_3^* we need $Q_3(x_3, y_3, z_3) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$. Having $Q_2(-\sqrt{3}, \sqrt{2}, 2)$, this means that

$$x_3^2 + y_3^2 - z_3^2 = 1 \quad \text{and} \quad -\sqrt{3}x_3 + \sqrt{2}y_3 - 2z_3 = 0. \tag{A.21}$$

Here, we set $x_3 = 0$. Then $y_3 = \sqrt{2}z_3$ and $z_3^2 = 1$. We choose $z_3 = 1$ and we finally get the point $Q_3(0, \sqrt{2}, 1)$.

Determination of $\mathbf{v}$. Since $Q_3^* \neq Q_3$ and $Q_3Q_3^* \nparallel \omega$, $Q_3Q_3^*$ gives a projection direction onto the plane ω. Furthermore, this direction is non-degenerate. Thus, we define $\mathbf{v}$ as in (1.25). In this way we get $Q_3' = Q_3^*$ (i.e., Q_3^* is $\pi_{\mathbf{v}}$–symmetric to Q_3) and $OQ_3 \parallel T_{\mathscr{H}}(Q_1')$, by (1.22).

Determination of P_1, P_2, P_3. Having fixed $\mathbf{v}$, we can project the points Q_1, Q_2, Q_3 into the plane ω. Setting $P_i = \Pi_{\mathbf{v}}(Q_i)$ $(1 \leq i \leq 3)$ we find the points:

$$P_1 = \Big(\tfrac{11}{10}, \tfrac{4\sqrt{6}-5\sqrt{3}}{10}, 0\Big), \quad P_2 = \Big(-\tfrac{2\sqrt{3}}{5}, \tfrac{5+\sqrt{2}}{5}, 0\Big), \quad P_3 = \Big(\tfrac{3\sqrt{3}}{10}, \tfrac{6\sqrt{2}+5}{10}, 0\Big) \tag{A.22}$$

and we can see that OP_1, OP_2, OP_3 are non-parallel. Finally, we note that

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{with coefficients} \quad h = \tfrac{5\sqrt{3}+3\sqrt{6}}{5+7\sqrt{2}},\ k = \tfrac{10+3\sqrt{2}}{5+7\sqrt{2}}. \tag{A.23}$$

As observed above in Ex. 1.12, the hyperbolic Pohlke's conic is an ellipse. In fact, according to Thm. 1.8, we find: $g(\frac{5\sqrt{3}+3\sqrt{6}}{5+7\sqrt{2}}, \frac{10+3\sqrt{2}}{5+7\sqrt{2}}) < 0$ and $f(\frac{5\sqrt{3}+3\sqrt{6}}{5+7\sqrt{2}}, \frac{10+3\sqrt{2}}{5+7\sqrt{2}}) < 0$. □

Ex. 1.13 (detailed calculations). To begin with, we set $\rho = 1$ and $Q_1(1, 1, 1)$.

Choice of Q_2. We look for $Q_2(x_2, y_2, z_2) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$. Since $Q_1(1, 1, 1)$, this is equivalent to require:

$$x_2^2 + y_2^2 - z_2^2 = 1 \quad \text{and} \quad x_2 + y_2 - z_2 = 0, \tag{A.24}$$

that is, $-2x_2y_2 = 1$ and $z_2 = x_2 + y_2$. For example, we can take $x_2 = 1$ and $y_2 = -\frac{1}{2}$. So we have the point $Q_2(1, -\frac{1}{2}, \frac{1}{2})$.

Choice of Q_3^*. We choose $Q_3^*(x_3^*, y_3^*, z_3^*)$ by imposing the same conditions as Q_2:

$$-2x_3^*y_3^* = 1, \quad z_3^* = x_3^* + y_3^*. \tag{A.25}$$

This time we can choose $x_3^* = \frac{3}{2}$ and $y_3^* = -\frac{1}{3}$. Thus, we have $Q_3^*(\frac{3}{2}, -\frac{1}{3}, \frac{7}{6})$.

Choice of Q_3. After Q_2, Q_3^* we need $Q_3(x_3, y_3, z_3) \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$. Having $Q_2(1, -\frac{1}{2}, \frac{1}{2})$, this means that

$$x_3^2 + y_3^2 - z_3^2 = 1 \quad \text{and} \quad x_3 - \tfrac{1}{2}y_3 - \tfrac{1}{2}z_3 = 0. \tag{A.26}$$

Equivalently, $-3x_3^2 + 4x_3y_3 = 1$ and $z_3 = 2x_3 - y_3$. Now, we choose $x_3 = \frac{5}{2}$. Then, $y_3 = \frac{79}{40}$ and we finally get $Q_3(\frac{5}{2}, \frac{79}{40}, \frac{121}{40})$.

Determination of $\mathbf{v}$. Since $Q_3^* \neq Q_3$ and $Q_3Q_3^* \nparallel \omega$, $Q_3Q_3^*$ gives a projection direction onto the plane ω. Furthermore, this direction is non-degenerate. Thus, we define $\mathbf{v}$ as in (1.26). In this way we get $Q_3' = Q_3^*$ (i.e., Q_3^* is $\pi_{\mathbf{v}}$–symmetric to Q_3) and $OQ_3 \parallel T_{\mathscr{H}}(Q_1')$, by (1.22).

Determination of P_1, P_2, P_3. Having fixed $\mathbf{v}$, we can project the points Q_1, Q_2, Q_3 into the plane ω. Setting $P_i = \Pi_{\mathbf{v}}(Q_i)$ $(1 \le i \le 3)$ we find the points:

$$P_1 = \left(\tfrac{103}{223}, -\tfrac{54}{223}, 0\right), \quad P_2 = \left(\tfrac{163}{223}, -\tfrac{250}{223}, 0\right), \quad P_3 = \left(\tfrac{389}{446}, -\tfrac{795}{446}, 0\right) \tag{A.27}$$

and we can see that OP_1, OP_2, OP_3 are non-parallel. Finally, we note that

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{with coefficients} \quad h = -\tfrac{145}{152},\ k = \tfrac{273}{152}. \tag{A.28}$$

As observed above in Ex. 1.13, the hyperbolic Pohlke's conic is a hyperbola. In fact, according to Thm. 1.8, we find: $g(-\frac{145}{152}, \frac{273}{152}) < 0$ and $f(-\frac{145}{152}, \frac{273}{152}) > 0$. □

Remark A.5 *After fixing $Q_1, Q_2 \in \mathscr{H}$ such that $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$, in the choice of points $Q_3^*, Q_3 \in \mathscr{H}$ of Ex. 1.12 and Ex. 1.13 the following facts are observed:*

a) *By Claim A.3, it cannot happen that $Q_3 = Q_3^*$. In fact, if $Q_3 = Q_3^*$ then we can get* (A.4) *renaming the points. More precisely, setting $R_1 = Q_3^*$, $R_2 = Q_1$ and $R_3 = Q_2$, we have*

$$OR_1 \parallel T_{\mathscr{H}}(R_2), \quad OR_2 \parallel T_{\mathscr{H}}(R_3), \quad OR_3 \parallel T_{\mathscr{H}}(R_1).$$

b) *After fixing $Q_1(x_1, y_1, z_1)$, $Q_2(x_2, y_2, z_2)$, we find that*

$$Q_3^* \in \mathcal{H}_1 = \mathscr{H} \cap \pi_1 \quad \textit{with} \quad \pi_1 : x_1x + y_1y - z_1z = 0 \tag{A.29}$$

and

$$Q_3 \in \mathcal{H}_2 = \mathscr{H} \cap \pi_2 \quad \textit{with} \quad \pi_2 : x_2x + y_2y - z_2z = 0. \tag{A.30}$$ [35]

Noting that $x_i^2 + y_i^2 - z_i^2 = 1$ $(i = 1, 2)$, from Claim 2.5 and a) *we deduce that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two disjoint hyperbolas with center at O. In particular, since $\pi_1, \pi_2 \nparallel \omega$, this means that there is no problem getting $Q_3Q_3^* \nparallel \omega$.*

Assuming $Q_3Q_3^* \nparallel \omega$ and setting $\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} = \overrightarrow{Q_3Q_3^*}$, it is easy to see that

$$l^2 + m^2 - n^2 = 2 - 2\left(x_3x_3^* + y_3y_3^* - z_3z_3^*\right). \tag{A.31}$$

Hence, $\mathbf{v}$ is degenerate if and only if $x_3x_3^* + y_3y_3^* - z_3z_3^* = 1$.

If $OQ_3 \nparallel OQ_1$, i.e., $Q_3 \neq \pm Q_1$, the plane

$$\pi(\lambda)\colon x_3x + y_3y - z_3z = \lambda \quad (\lambda \in \mathbb{R}) \tag{A.32}$$

is not parallel to π_1 and $\pi(\lambda) \cap \mathcal{H}_1 \neq \emptyset$ if $|\lambda|$ is large enough. So, if we already have Q_3 and $OQ_3 \nparallel OQ_1$, we can choose $Q_3^* \in \mathcal{H}_1$ so that $l^2 + m^2 - n^2$ takes any large enough positive or negative value. [36]

[35] Note that $\pi_1 \nparallel \pi_2$ because $OQ_1 \nparallel OQ_2$.

[36] Just note that we can take coordinates $\bar{x}, \bar{y}$ in π_1 such that $\mathcal{H}_1$ has equation $\bar{x}\bar{y} = 1$. In the same coordinates the straight line $\pi_1 \cap \pi(\lambda)$ has equation $a\bar{x} + b\bar{y} = \alpha\lambda$ for suitable a, b (not both zero) and $\alpha \neq 0$.

c) *Since* $R, S \in \mathscr{H}$ *and* $OR \parallel T_{\mathscr{H}}(S) \Rightarrow OR \nparallel OS$, *we necessarily have*

$$OQ_2 \nparallel OQ_1, \quad OQ_3^* \nparallel OQ_1, \quad \text{and} \quad OQ_3 \nparallel OQ_2. \tag{A.33}$$

On the other hand, we can even choose $Q_3 = \pm Q_1$ *and* $Q_3^* = \pm Q_2$.

To prevent two of the segments OP_1, OP_2, OP_3 *from being parallel, it is clear we must have* $OQ_3 \nparallel OQ_1$, *because* $OQ_3 \parallel OQ_1 \Rightarrow OP_3 \parallel OP_1$. *Then, after choosing* Q_2 *and* Q_3, *with* $OQ_3 \nparallel OQ_1$, *it is easy to see that:*

$$OP_1 \parallel OP_2 \;\Leftrightarrow\; Q_3Q_3^* \parallel \langle O, Q_1, Q_2\rangle, \tag{A.34}$$
$$OP_1 \parallel OP_3 \;\Leftrightarrow\; Q_3Q_3^* \parallel \langle O, Q_1, Q_3\rangle, \tag{A.35}$$
$$OP_2 \parallel OP_3 \;\Leftrightarrow\; Q_3Q_3^* \parallel \langle O, Q_2, Q_3\rangle, \tag{A.36}$$

where with $\langle O, Q_i, Q_j\rangle$ *we indicate the plane through* O, Q_i, Q_j.

Starting from the last, we see that the second of (A.36) *holds iff*

$$\overrightarrow{OQ_3^*} = \lambda \overrightarrow{OQ_2} + \mu \overrightarrow{OQ_3}, \tag{A.37}$$

for suitable λ, μ. *Having* $OQ_2, OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$ *and (by Claim A.3)* $OQ_3 \nparallel T_{\mathscr{H}}(Q_1)$, *it follows that* $\mu = 0$. *Therefore, we deduce that the second condition of* (A.36) *holds iff* $OQ_3^* \parallel OQ_2$, *i.e.,* $Q_3^* = \pm Q_2$.

Similarly, we see that the second of (A.35) *holds iff*

$$\overrightarrow{OQ_3^*} = \lambda \overrightarrow{OQ_1} + \mu \overrightarrow{OQ_3}, \tag{A.38}$$

for suitable λ, μ. *Here* $OQ_1, OQ_3 \parallel T_{\mathscr{H}}(Q_2)$ *and* (A.38) *immediately give* $OQ_3^* \parallel T_{\mathscr{H}}(Q_2)$, *i.e.,* $OQ_2 \parallel T_{\mathscr{H}}(Q_3^*)$. *This contradicts Claim A.3, because* $OQ_3^* \parallel T_{\mathscr{H}}(Q_1)$. *We conclude that the second of* (A.35) *never holds.*

Finally, the second of (A.34) *holds iff, for suitable* λ, μ,

$$\overrightarrow{OQ_3^*} = \overrightarrow{OQ_3} + \lambda \overrightarrow{OQ_1} + \mu \overrightarrow{OQ_2}. \tag{A.39}$$

Now, $OQ_2 \parallel T_{\mathscr{H}}(Q_1)$ *and* (A.39) *imply that* $OQ_3^* \parallel T_{\mathscr{H}}(Q_1) \Leftrightarrow \lambda + x_1x_3 + y_1y_3 - z_1z_3 = 0$. *Furthermore, since* $OQ_1, OQ_3 \parallel T_{\mathscr{H}}(Q_2)$, *if* (A.39) *holds then* $Q_3^* \in \mathscr{H}$ *iff*

$$1 + \lambda^2 + \mu^2 + 2\lambda\big(x_1x_3 + y_1y_3 - z_1z_3\big) = 1. \tag{A.40}$$

In conclusion, we easily see that (A.39) *holds iff*

$$\overrightarrow{OQ_3^*} = \overrightarrow{OQ_3} - \bar\lambda \overrightarrow{OQ_1} \pm \bar\lambda \overrightarrow{OQ_2}, \tag{A.41}$$

with $\bar\lambda \overset{\text{def}}{=} x_1x_3 + y_1y_3 - z_1z_3$.

Summarizing up, after fixing Q_1, Q_2 *and* Q_3, *with* $OQ_3 \nparallel OQ_1$, *the segments* OP_1, OP_2, OP_3 *are non-parallel if and only if* Q_3^* *satisfies the conditions* (1.23).

d) *It is easy to see that* O, Q_1, Q_2, Q_3 *are coplanar if and only if* $Q_3 = \pm Q_1$. *Indeed, if* $\overrightarrow{OQ_3} = \lambda \overrightarrow{OQ_1} + \mu \overrightarrow{OQ_2}$, *then the condition* $OQ_2 \parallel T_{\mathscr{H}}(Q_3)$ *gives*

$$x_2(\lambda x_1 + \mu x_2) + y_2(\lambda y_1 + \mu y_2) - z_2(\lambda z_1 + \mu z_2) = 0. \tag{A.42}$$

Since $x_1x_2 + y_1y_2 - z_1z_2 = 0$ *and* $x_2^2 + y_2^2 - z_2^2 = 1$, (A.42) *implies* $\mu = 0$. *Then*

$$Q_3 \in \mathscr{H} \quad \text{and} \quad \overrightarrow{OQ_3} = \lambda \overrightarrow{OQ_1} \quad \Rightarrow \quad \lambda = \pm 1. \tag{A.43}$$

Similarly, the points O, Q_1, Q_2, Q_3^* *are coplanar if and only if* $Q_3^* = \pm Q_2$.

We may conclude that the points O, Q_1, Q_2, Q_3, Q_3^* *are coplanar (i.e.,* O, P_1, P_2, P_3 *are ultimately collinear) if and only if we choose* $Q_3 = \pm Q_1$ *and* $Q_3^* = \pm Q_2$.

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